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Gödel and Beyond: A Treatise on the Notion of Indeterminancy

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A Treatise on the Notion of Indeterminacy

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Senior Honors Thesis

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Introduction

For my Honors thesis in Mathematics, I have had the opportunity to reflect on the notions of indeterminacy and incompleteness within various field associated with mathematics. I initially gained a reading comprehension on paradoxes, emphasizing Russell's paradox of the library. I then studied first order logic, using propositional calculus as a means of examining a formal system which is consistent and complete for the purpose of comparison and future reference. I explored various proofs utilizing the essence of Cantor's diagonal argument, which allowed me the opportunity to better comprehend the ideas of infinity, contradiction, incompleteness, and indeterminacy. All this led to a Halting version proof of Gödel's Incomplete Theorem. I examined the proofs of Gödel's two incompleteness theorems and concluded with a discussion on Chaitin's Omega.

Zeno's Paradox

The origin of my mathematics senior honors thesis can be traced to Douglas Hofstadter's novel, Gödel, Escher, Bach: An Eternal Golden Braid. In the opening dialogue, Hofstadter introduced Zeno's paradox of Tortoise and Achilles. In this paradox, Tortoise started by challenging Achilles to a foot race in which Tortoise claimed that he would always win when given a head start of any distance. Tortoise's reasoning was that if he is given a head start, within the time Achilles has covered the head start distance, Tortoise would have traveled some distance more in that time. Within the time Achilles covered the distance to the next point, Tortoise would have traveled yet a little further on in that time. This reasoning was repeated over and over again; every time Achilles reaches a spot where Tortoise had passed, Tortoise would have moved on a little more in the time it took Achilles to reach that spot, so Achilles would never be able to catch up with Tortoise in this race. Presented with this reasoning, Achilles decided not to race Tortoise for fear of losing the race.

Now common sense should incline you believe that Achilles is victorious in any foot race against a slow Tortoise, but how can one refute the reasoning presented above? It makes absolute sense, yet the conclusion that Tortoise wins the race is illogical. This paradox alerts the possibility that our human ability to reason is inconsistent with the results of reality or maybe our ability to comprehend this situation is incomplete. Fortunately, the solution to this paradox is to understand that time and space may be infinitely divisible but the sum of some infinite series is finite. For example, the sum of the infinite series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots..$ equals 1, a finite term. Say you want to cross a room that is four meters wide. It takes you four seconds to cover two meters, then it takes you another two second to cover one meter, one second to cover half a meter, half a second to cover a quarter of a meter and so on. The sum of all the distance covered is four meters and it takes a total of eight seconds, both finite terms.

Gödel, Escher, Bach

A meta-level above the story of Achilles, Tortoise, and Zeno concerns Kurt Gödel, M.C. Escher, and Johann Sebastian Bach. Hofstadter presents the works of Gödel, Escher and Bach, to show examples of loops like mathematical recursion, a staircase, and a musical scale. The meta-level above Gödel, Escher, and Bach, which connects them all, are the existence of these loops in the field of mathematics, art and music. A specific example of loops is the Strange

Loop, a common occurrence in many of Escher's works. The definition of a Strange Loop is "a phenomenon in which, whenever movement is made upwards or downwards through the levels of some hierarchical system, the system unexpectedly arrives back where it started."[1] Escher's works with the Strange Loop include *Waterfall*, *Ascending and Descending*, *Drawing Hands*, and *Print Gallery*. While these works are ascetically pleasing to the eye and momentarily spark the interest of the mind, Hofstadter offers a deeper insight on the significance of Strange Loops. "Implicit in the concept of Strange Loops is the concept of infinity, since what else is a loop but a way of representing an endless process in a finite way?"[2] Similar to Zeno's paradox, the conflict between the finite and infinity arises. In mathematics, Gödel is able to apply the concept of a Strange Loop in his proof of his First Incompleteness Theorem. As we are able to talk about language in language through self reference, Gödel's application to mathematics and number is original and unique, yielding many consequences like the incompleteness of any formal system.

First Order Logic

The purpose of approaching first order logic via propositional calculus is to examine a formal system which is complete and consistent. In propositional calculus, there exists a set of axioms, the rules of natural deduction in logic, from which all statements are deducible from the axioms. The focus is on the set of well-formed formulas (wffs), valid statements derived from the axioms, whose statements are tautologies. It is equally important to distinguish the differences between truth and validity. Given the rules of inference, proofs are derived when the rules are properly applied. Proofs are valid if no errors have been made in the application of these rules. Also, formulas are 'well-formed' (wff) if they are syntactically correct.

There are two types of arguments in propositional calculus, truth tables and rules of inference. At this point, it is possible to map the symbols: P, Q, and etc, representing atomic propositions to the Boolean variable with values T & F, representing true and false. Using truth tables to interpret the logical connectives, wffs become functions, and acquire the values T & F. A proposition (wff) that yields T in all possible cases is called a tautology. A contradiction is any wff S of the form in which S and ~S are both theorems in the system. With truth tables, a theorem is any wff provable from no assumptions, so it is derived from the rules of deduction which are expressed as axioms. Then one can say that the theorem follows from the rules of

[1] Eric W. Weisstein. "Strange Loop." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/StrangeLoop.html
[2] Hofstadter, Douglas R. Gödel, Escher, Bach. New York, Vintage Books, pp 15, 1979

inference. Now it must be determined if the implicit 'meanings' deducted from the rules of inference match in some way the 'meaning' provided by the truth table. If the 'meanings' match, then this justifies the use of the same symbols in both the truth tables and in the rules of proof. If they match, one can interpret the logic of proof in terms of truth. Should all theorems be tautologies, then the system is consistent; and should all tautologies be theorems, then it is complete. Now consistency and then completeness is achieved if can established that there is a one to one correspondence between the tautologous wffs and the derivable wffs.[3] Propositional calculus is complete since it can be proven that all theorems are tautologies with truth table representation and all tautologous wffs are theorems, derived from the axioms of the system.[4]

Paradox of the Library

Suppose you are in a college library with 100 books. The librarian is having trouble remembering the names of all the books in the library so the librarian decides to create a catalogue which will list all the books in the library. This catalogue contains 100 entries for the 100 books in the library. Now the librarian is very picky and likes to keep track of every book in the library, so the dilemma is whether the library now has only 100 books or 101 books, counting the catalogue.

Now consider all the college libraries in the United States. A group of librarians in the universities is very picky and insists that the catalogues of all books in the library must list the catalogue itself, so we call these catalogues the fat catalogues. The other group of librarians believes strongly that the catalogues themselves should not be listed in the catalogue of the books in the library. So we call the catalogues which do not list themselves the thin catalogues.

Now suppose you are at the Library of Congress. Being the Library of Congress, it must account for every catalogue of every college in the nation. There exist two different types of catalogues in the colleges, the fat catalogues and the thin catalogues, so the librarian at the Library of Congress must create two separate catalogues to account for all the catalogues. The goal is to make each catalogue complete so that all catalogues of each type is listed. The first catalogue is labeled Fat A Catalogue and includes all the fat catalogues in the colleges; fat catalogues are catalogues which list itself. The second catalogue is labeled Thin B Catalogue

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[3] Lemmon, E. J. Beginning Logic. Indianapolis, Hackett Publishing Company, 1978
[4] Bloxham, Mike. E-mail to Samuel Wu, 15 April 2006.

and includes all the thin catalogues in the colleges; thin catalogues are catalogues which do not list themselves.

The Library of Congress is able to finish Fat A catalogues without any complications. In addition to listing all the fat catalogues from the libraries, the last entry in the catalogues is Fat A Catalogue itself. Moving on to the Thin B Catalogue, the librarian almost finishes the catalogue when he faces a 'situation'.

The goal from the start is to create a complete catalogue of thin catalogues, labeled as Thin B Catalogue. The situation the librarian faces is trying to answer the question: How to create a complete catalogue of thin catalogues?

The dilemma is that since Thin B Catalogue is a catalogue of thin catalogues, since it does not list itself, by rule the last entry in Thin B Catalogue should be 'Thin B Catalogue' itself. However if this occurs, Thin B Catalogue no longer exists as a catalogue of catalogues which does not list itself; instead it would be a fat catalogue. On the other hand, if Thin B Catalogue does not contain itself, then by definition, Thin B Catalogue is a catalogue which does not list itself and Thin B Catalogue should list itself. In essence, the catalogue of thin catalogues at the Library of Congress is not complete since Thin B Catalogue is not listed anywhere and no one can determine where it should be listed.

Russell's Paradox

This paradox is known as Russell's paradox. It challenges our basic understanding of sets. A set is defined by Cantor as a collection of objects of thought. In the late 1800's, Gottlob Frege, a German mathematician, tries to formalize arithmetic and begins with the axiom of a collection, a set. Membership in a set is determined based on the axioms. "Frege also wanted to be able to define the natural numbers in purely logical terms. In general, the number of elements in a given set can be defined to be the collection of all those sets that can be matched up one-to-one with the given set."[5] Thus, Frege's arithmetic makes use of sets of sets. In 1902, Bertrand Russell writes a letter to Frege explaining why reasoning with sets of sets can easily lead to contradictions.

Russell's paradox is formally stated as: "Let R be the set of all sets which are not members of themselves. Then R is neither a member of itself nor not a member of itself.

[5] Davis, Martin. Engines of Logic. New York, W.W. Norton & Co., pp. 55, 2000

Symbolically, let $R = \{x : x \notin x\}$. Then $R \in R$ iff $R \notin R$."[6] Applying the logic above, if R is a member in this set then it would no longer be the sets of all sets which are not members of themselves. But if R is not in this set then by rule as a property of R, it should be a member of this set.

Russell introduces the idea of *theory of types* as his response to this paradox. This theory ranks the objects within sets in varying levels of hierarchy. It is then "possible to refer to all objects for which a given condition (or predicate) holds only if they are all at the same level or of the same 'type'."[7] This version of the solution is based on the *vicious cycle principle*. This principle states that "before a function can be defined, one first has to specify exactly those objects to which the function will apply."[8] So in the formal version of Russell's paradox, the set R is not an object of that set. Similarly in the Library of Congress, Thin B Catalogue is not applicable to be catalogued by the Library of Congress. Or else, considering the paradox of the library under the vicious cycle principle, the Library of Congress can create a meta-catalogue Thin C Catalogue which includes all thin catalogues and Thin B Catalogue. It then creates another meta-catalogue Thin D Catalogue which includes all thin catalogues, Thin B Catalogue and Thin C Catalogue. This action repeats in a never ending cycle. Ultimately, this solution did not find favor.

Theorem: Cardinality of the Power Set of Set S

A theorem is now introduced with the primary focus on the proof of this theorem. It employs the basic logic of reasoning as introduced above.

Theorem A: The cardinality[9] of the power set of *S* is always strictly greater than the cardinality of *S* for a set *S*.

[6] Eric W. Weisstein. "Russell's Antinomy." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/RussellsAntinomy.html

[7] Irvine, A. D., "Russell's Paradox", *The Stanford Encyclopedia of Philosophy (Summer 2004 Edition)*, Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2004/entries/russell-paradox/>

[8] Irvine, A. D., "Russell's Paradox", *The Stanford Encyclopedia of Philosophy (Summer 2004 Edition)*, Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2004/entries/russell-paradox/>

[9] Cardinality is defined as the number of elements in a set, denoted | * |

Proving this for an elementary finite set is easy by means of an example. Consider the set $S = \{1\ 2\ 3\}$, the set S has three elements so the cardinality of S is three. A power set is the set of all subsets of the set. So the power set of S, denoted $P(S)$, is the set $P(S) = \{\ \{\ \}\ \{1\}\ \{2\}\ \{3\}\ \{1\ 2\}\ \{1\ 3\}\ \{2\ 3\}\ \{1\ 2\ 3\}\ \}$. Referring to this example, it is clear that $|S| = 3$ and $|P(S)| = 8$. In fact, the order of a power set of a set of order n is 2^n.[10] A set of order n has n elements; to generate the power set, each subset has the option of either containing the element or not containing the element. Whether each element is included is independent of each other, so there are 2^n subsets which forms the power set.

Note: In Theorem A, S may be an infinite set.

Proof:[11]

Suppose the sets S and $P(S)$ have the same cardinality. The purpose of introducing the term cardinality is to give meaning to 'same number' in the case of infinite sets. By definition of cardinality, there is some one-to-one correspondence between S and $P(S)$ so under this one-to-one correspondence, there exists a one-to-one mapping for each element a of the set S ($a \in S$) to a corresponding element $R(a)$ in the set $P(S)$, i.e. ($R(a) \in P(S)$). Sometimes the set $R(a)$ contains a as a member of itself and sometimes it does not. Consider the collection of all the elements $a \in S$ for which $R(a)$ does not contain a. The collection of all these elements is some particular subset Q of S. Under the assumption of the one-to-one correspondence, we have $Q = R(q)$, for some q in S. Now ask, 'Is q in Q or is it not?' First suppose it is not. Then q belongs to the collections of elements of S that is included in Q, so q must belong to Q: a contradiction. The alternative supposition is that q is in Q. But q cannot belong to Q or else Q is no longer the collection of all elements $a \in S$ for which $R(a)$ does not contain a. So q does not belong to Q: again a contradiction. Therefore the assumed one-to-one correspondence between S and $P(S)$ cannot exist and S and $P(S)$ do not have the same cardinality. Also, since we are able create a one-to-one correspondence to map all the elements $a \in S$ to a particular $R(a)$ that

[10] Eric W. Weisstein. "Power Set." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/PowerSet.html
[11] Penrose, Roger. The Road to Reality. New York, Alfred A. Knopf, pp 369, 2004

contains a but establish the element Q in $P(S)$ which has no corresponding element in S, then the cardinality of $P(S)$ is greater than the cardinality of S.

Visually:

Elements of S	Elements of the $P(S)$
a	$\{a, b\}$
b	$\{m, n\}$
c	$\{c, f, r\}$
d	$\{r, t, s\}$
f	$\{z, w, u, i\}$
.	.
.	.
q	$Q = \{b\, d\, f \ldots\ldots\}$
.	.
.	.

Q is the collection of elements $a \in S$ for which $R(a)$ does not contain a.
Question: 'Is q in Q or not?'

This theorem and proof proves the existence of higher orders of infinity. When S is an infinite set, |S| is infinite. Then $P(S)$ is also an infinite set and $|P(S)|$ is infinite since S is an infinite set. But it is trivial that S can be put in a one-to-one correspondence with a proper subset of $P(S)$ so by Theorem A, the cardinality of $P(S)$ exceeds the cardinality of S. Thus there exist higher orders of infinity.

Halting Problem

Is there a general way of determining whether a program, when given a particular input, will complete its computation and halt? In 1936, Alan Turing proves that a general algorithm to solve the halting problem for all possible inputs cannot exist.

For the proof, it is necessary to list programs and call the n[th] program f_n. For example
f_1 = Compute 2x for $x \in \mathbb{Z}$.
f_2 = State the largest prime number.
f_3 = State the last digit of Pi.
I will leave it to the intuition of the reader to determine that f_1 halts but f_2 and f_3 do not halt.

Theorem B: In a consistent system, no general algorithm exists to decide the halting of all programs without running them.

Proof:

Suppose that it is possible to enumerate all programs where the n[th] program is f_n. Enumerate means to number the programs in a list in any arbitrary order. While there may be infinitely many programs, each program is restricted to a finite size and limited by a finite number of symbols, e.g. letters and numbers. Assume there is no one program that is infinitely long, so the enumerated programs are all of finite length. Now provided that we can tell whether a program halts or not, then we can construct the following program, which we shall call H. Program H take $n \in \mathbb{Z}$ as inputs. Let H(n) be defined as the program:

$$H(n) = \begin{cases} f_n(n)+1 & iff \quad f_n(n) \quad halts \\ -99 & iff \quad not \end{cases} \Big\}$$

Now suppose $H(\ )$ is the h[th] program on the list of enumerated programs and try to compute $H(h)$. Note that $H(h) = f_h(h)$ since it is itself the h[th] program. First assume that $H(h)$ halts; then $f_h(h) = f_h(h)+1$. Assuming the system is consistent, $f_h(h) \neq f_h(h)+1$. So the conclusion is that $H(h)$ does not halt. H is presumed to be able to detect that it does not halt and return the value -99, which is to say $H(h)$ halts. The fact that $H(h)$ even returned an output is contradictory since it is a program that does not halt. So the original assumption that the program $H(h)$ exists which relies on the ability to tell whether a program halts is false and a general program that is able to do so cannot exist.

Richard's Paradox

Now Richard's paradox magnifies the consequences of self-reference. Using ordinary English, expressions that define or specify properties of natural numbers can be created. These expressions are composed of words which are created from a finite list of letters and as a requirement, each expression is to be of finite length. There are infinitely many expressions as there are endless things to say about natural numbers.

To continue with the paradox, enumerate these expressions of finite length such that: E(1) is the first expression, E(2) is the second, E(3) is the third, E(n) is the nth expression, and so on to infinity.[12] For example, for $x \in \aleph$, define E(5) as x is a prime number and E(8) as x is divisible by 2. Evaluating these expressions, for x = 11, E(5) is true and E(7) is false; for x = 12, E(5) is false and E(7) is true. Now somewhere in this list is the expression: 'for any number r, E(r) is not true of r.' Suppose this expression is in the q^{th} position on this list so that E(q) states: 'for any number r, E(r) is not true of r.'

Now for any number n, determine whether the nth expression is true. Referring to our examples, for x = 5, E(5) is true; for x = 7, E(7) is false. Similarly, try to determine if E(q) is true when r = q. Assume that E(q) is true of q. It follows that E(r) is not true of r, which in this case means that E(q) is not true of q: a contradiction. Suppose E(q) is not true of q; then E(r) is not true of r, meaning E(q) is true of q: a contradiction. In conclusion, the number r is only true of the r^{th} expression on the list if and only if it is not true for the r^{th} expression on the list.

The illogical conclusion of Richard's paradox is the result of the use of self-reference with language. This result is refuted by the requirement that the language and meta-language are to be kept separate. Be careful to distinguish expressions that define properties of natural numbers and the statements which define expressions that define properties of natural numbers. These statements which define expressions that define properties of natural numbers belong in the meta-language, not to be included in the list.[13] Gödel's stoke of genius is the intuition to map these expressions [the language] into numbers, where there exists only one level of numbers based on the Gödel numbering system.

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[12] Berlinski, David. The Advent of the Algorithm. New York: Hartcourt, Inc. 2000 pp 118-120
[13] Wikipedia. "Richard's paradox" From http://en.wikipedia.org/wiki/Richard's_paradox

Definitions

Incompleteness – "A formal theory is said to be incomplete if it contains fewer theorems than would be possible while still retaining consistency."[14]

Consistency – The impossibility to prove that a statement and its negation are both true, the absence of contradiction.[15]

Undecidable – "Not decidable as a result of being neither formally provable nor unprovable."[16]

Theorem – A proposition that can be proven on the basis of explicit assumptions.[17] "A mathematical statement that has been proven is called a theorem."[18]

Gödel's First Incompleteness Theorem

Formally, Gödel's theorem states, "To every ω-consistent recursive class κ of formulas, there correspond recursive class-signs r such that neither (v Gen r) nor Neg(v Gen r) belongs to Flg(κ), where v is the free variable of r."[19]

Informally, Gödel's incompleteness theorem as, "All consistent axiomatic formulations of number theory include undecidable propositions."[20]

Theorem C (restated): In any consistent formal system containing an axiomatic formulation of number theory, it is possible to construct a statement that is true but not provable in the system.

Proof:

Assume the system is consistent. Let G be the statement "This statement cannot be proven in this system." Suppose G can be proven in this system; then G is a theorem of this system. However, the theorem of a consistent system will always express a tautology. So if G is

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[14] Eric W. Weisstein. "Incompleteness." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Incompleteness.html
[15] Eric W. Weisstein. "Consistency." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Consistency.html
[16] Eric W. Weisstein. "Undecidable." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Undecidable.html
[17] *The American Heritage® Stedman's Medical Dictionary, Copyright © 2002, 2001, 1995 by Houghton Mifflin Company. Published by Houghton Mifflin Company.* From http://www.dictionary.com
[18] Eric W. Weisstein. "Proof." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Proof.html
[19] Gödel, Kurt. "Über Formal Unentscheidbare Sätze der *Principia Mathematica* und Verwandter Systeme, I." *Monatshefte für Math. u. Physik* pp. 38, 173-198. 1931
[20] Hofstadter, Douglas. R. Gödel, Escher, Bach: An Eternal Golden Braid. New York: Vintage Books, p. 17, 1989

a theorem, then it is stating an inconsistency. Since a theorem cannot state a inconsistency, deduce that G is not a theorem of the system i.e. G cannot be proven in this system. The conclusion that G cannot be proven means that the statement G, "This statement cannot be proven in this system", is true. Since the statement G is a truth that cannot be proven in the system, the system is incomplete.

As was to be shown, G is a statement which is true but cannot be proven in the system.

Gödel's Second Incompleteness Theorem

Theorem D: Any formal arithmetic system which assumes its own consistency can prove its own consistency if and only if it is inconsistent.[21]

Proof:

($\Rightarrow$) Assume Formal Arithmetic is Consistent, denoted CONS. As a result of Gödel's first incompleteness theorem, if the system is consistent, then the statement G must be true; CONS $\Rightarrow$ G. Reading the assertion sign '$\vdash$' as 'therefore'; note that when no antecedent appear to the left of the symbol '$\vdash$' , the line is to be read as 'it follows from the axioms, or equivalently, from the rules of inference.

$\vdash$ CONS $\Rightarrow$ G	(First Incompleteness Theorem)
$\vdash$ CONS	(Assume can prove its own consistency)
$\vdash$ G	(MPP)

Modus ponendo ponens (MPP) "is the principle that, if a conditional holds and also its antecedent, then its consequent holds."[22] Given CONS and CONS $\Rightarrow$ G, conclude G can be proven. But G states that it cannot be proven, hence a contradiction exists. So the system can only prove its own consistency if the system is inconsistent.

($\Leftarrow$) Assume Formal Arithmetic is Inconsistent. If the system is inconsistent, then it follows that the system is also consistent.

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[21] Eric W. Weisstein. "Gödel's Incompleteness Theorem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/GoedelsIncompletenessTheorem.html

[22] Lemmon, E.J. Beginning Logic. Indianapolis: Hackett Publishing, pp 61, 1978

Gödel and Turing to Omega

The result of Gödel's Incompleteness theorem implies that given a consistent system in mathematics, there exist true statements which cannot be proven from the axioms and rules of inference in the system. Turing shows that there is no general algorithm for deciding if an arbitrary program will halt or not. Meaning, "There is no consistent system of axioms strong enough to decide the halting of all programs without running them."[23] The conclusions of both Gödel and Turing infer that in consistent systems, there is a sentence which cannot be proven. Similar to the concept of incompleteness, Gregory Chaitin proposes the existence of a number Omega, denoted **Ω**, which is the probability that an arbitrary program will halt. From the halting problem and the need to hypothetically run every program up to a size of n bits to determine whether it halts or not, Chaitin proposes the existence of the random number Omega.

Omega

Properties of Omega are that it is random, incompressible, irrational, normal and transcendental. Transcendental numbers are numbers that are not the roots of any integer polynomial, meaning it is not an algebraic number of any degree.[24] An algebraic number is any number which is the root of an integer polynomial. It follows that transcendental numbers are irrational numbers since all rational numbers are algebraic numbers of the first degree. Omega is normal since there is a random distribution in which all digits between zero and one are equally likely to appear.[25] A number is said to be random if the shortest program to compute the number has as many bits as the number itself.[26] Note that this is definition of randomness is also the definition of incompressibility. A non-random number is a number which can be computed by a program with considerably fewer bits than the number itself.[27] Unlike numbers like Pi or the square root of two, which are irrational and normal, it is possible to state the next digit in the sequence. This is not the case for Omega.

The Omega number is important in that if the first few thousand digits of the number are known, theoretically it would provide an answer to most of the open questions in mathematics,

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[23] Bennett, Charles. "On random and hard-to-describe numbers: Chaitin's Omega."
[24] Eric W. Weisstein. "Transcendental Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/TranscendentalNumber.html
[25] Wikipedia. "Normal number." http://en.wikipedia.org/wiki/Normal_number
[26] Bennett, Charles. "On random and hard-to-describe numbers: Chaitin's Omega."
[27] Chaitin, Gregory. "Meta Math." http://www.arxiv.org/PS_cache/math/pdf/0404/0404335.pdf

including propositions that if false could be refuted in a finite number of steps. Examples include Fermat's last theorem or Goldbach's conjecture which asserts that every even number is the sum of two primes. This is because "finitely refutable conjectures are equivalent to the assertion that some computer program that searches systematically for the allegedly nonexistent object will never halt."[28] This draws a correspondence between mathematical statements and the sequence of '0's and '1's which make up Omega. In essence, each binary digit is a theorem. For Fermat's last theorem, which states that the equation $x^n+y^n=z^n$ has no solution in the positive integers when n is greater than 2, can be simply refuted by a single finite counterexample. That is a set of integers x, y, z, and n that solve the equation, this correspondence to a program which will never halt. Before Andrew Wiles' proof of Fermat's Last Theorem in 1993, this argument served to be a theoretical approach to comprehending the theorem's proof. Now its relation to Omega is that **Ω** is generated by an arbitrary computer which takes programs as inputs and is able to determine whether it halts or not. Ω_n is the overall halting probability for all programs of n bits or fewer. Ω_n is computed by a systematic but unending search for all programs that halt by running first bit of all programs for one second, then the second bit of the programs for two seconds, then the third bit for three seconds, and so on until all programs less than n bits have halted. Assuming the programs are pre-fix free, every time a program of m-bits halts, $(\frac{1}{2})^m$ is added to Ω_n, for $m \in Z$. Once Ω_n is computed, no more programs of length n or less can halt, else if a program of length n or less does halt, it would imply that one of the known digits of **Ω** is altered.[29] This is guaranteed from the result of Turing's halting problem.

Referring back to finitely refutable conjectures like Fermat's last theorem or Goldbach's conjecture, if it were possible to precisely estimate **Ω** say for the first 5,000 bits, then **Ω** would verify the results of Fermat's last theorem, Goldbach's conjecture and all other finite refutable conjectures. The ability to code these conjectures into 5,000 bits or less, but never more than 5,000, means that somewhere in Ω_n is the proof of a program which did not find a x, y, z and n which solves the equation $x^n+y^n=z^n$ for when n is greater than two. Similarly a program did not find an even number which is not the sum of two primes. The results of Gödel and Turning assure that Ω_n cannot determine the halting probability for programs greater than length n. The first n bits of **Ω** solve the halting problem for all programs of n bits or fewer; this proves the incompressibility of all incompressible numbers n bits or fewer. If Ω_n is computed by a program

[28] Bennett, Charles. "On random and hard-to-describe numbers: Chaitin's Omega."
[29] Chaitin, Gregory. "The Limits of Reason." Scientific America. March 2006

shorter than n bits, then that program has just computed the first incompressible n-bit integer and it must be incompressible itself, which is a contradiction.

Conclusion

Gödel and Turning effectively answered David Hilbert's second question, which asked whether axioms of logic can be proven consistent. The answer is no. Chaitin's Omega reaffirms this idea of incompleteness since Ω_n is constrained by the *n* bits by which it is computed and does not give insight to numbers greater than *n* bits. The significance of this notion of Omega is the idea that "no single system of axioms will suffice for understanding (or compressing) mathematics."[30] The inability to prove everything in mathematics allows room for new ideas and creativity, showing that math cannot be completely axiomatic.

[30] Chaitin, Gregory. "Meta Math." http://www.arxiv.org/PS_cache/math/pdf/0404/0404335.pdf

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