Financial Mathematics: Options, Arbitrage, and the Black-Scholes PDE

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Financial Mathematics:
Options, Arbitrage, and the Black-Scholes PDE

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1 Abstract

This paper aims to derive and solve the Black-Scholes partial differential equation (PDE) used to price options. Options allow investors the possibility for great gain with small probability for a large loss. Thus, they can be very valuable if money is invested correctly. Brownian motion can be used to model the change in stock prices and is the jumping off point for the derivation of the Black-Scholes PDE. The derivation of the PDE and its solution using the method of Fourier transforms will be shown. An application of the pricing formula using actual stock values will also be given.

2 Introduction to Options and Arbitrage

In its simplest form, an option is the right, but not the obligation, to buy or sell a certain security (in our case a stock) at an agreed upon time in the future at an agreed upon price. The agreed upon price is called the striking price and the date is called the strike time or expiry date. To distinguish between buying the right to sell versus the right to buy a stock we define a call option as the right to buy a security in the future and a put option as the right to sell a security in the future. An option that can only be exercised at maturity is known as a European option, and an option that can be exercised at any time, up to and including the expiry date, is known as an American option. Mathematically, a European option is easier to deal with, but American options are more common in the real world.

In the financial world it is possible to sell an asset before a person actually owns it. Borrowing and selling a security with the agreement to buy it later is called adopting a short position in the object. If someone purchases an object first and then sells it in the future they are said to be adopting a long position. Adopting a short position can be beneficial to an investor who is concerned about a decrease in price of an object.

In its basic form, arbitrage exists whenever two financial instruments are mispriced relative to one another. Consider an experiment that has \( m \) possible outcomes. Then the Arbitrage Theorem states that the probabilities for the \( m \) outcomes are such that for each bet the expected payoff is zero, or there exists some bettering strategy for which the payoff is positive regardless of the outcome of the experiment. When the probabilities are such that the expected payoff is zero, we say the situation is arbitrage-free. If the payoff
is positive regardless of the experiment’s outcome, then arbitrage exists.

3 Brownian Motion

To understand the idea behind a random walk, we start with an example. Suppose someone flips a coin. If the coin lands on heads, the person takes a step to the right (the positive direction), and if the coin lands on tails the person takes one step to the left (negative direction). If we continue on in this manner, the evolution of this process is called a random walk. A simulation of 100 coin flips in Excel yielded the following graph. (Note: Instead of starting at 0, the assumed starting position is 10. Think of this as starting at 10th Street, and then walking north one block if the coin lands on heads, and south one block if the coin lands on tails. The graph then shows this person’s progress at 1 unit time intervals.)

The coin example describes a discrete random walk. By using the central limit theorem we can find an analogous way of describing continuous random motion. Its mathematical model is called Brownian motion, named after Scottish Botanist Robert Brown who first described random motion in terms of pollen particles suspended in a liquid. The time evolution for this motion is now described as a stochastic process and denoted $S(t)$. The mathematical description of Brownian motion was developed by Norbert Wiener, and thus the stochastic process is often called a Wiener process and is denoted $W(t)$. 
This then leads to the following stochastic differential equation known as a generalized Wiener process

\[ dZ = \mu dt + \sigma dW(t) \]  

(1)

where \( \mu \) is the drift and \( \sigma \) is known as the volatility. If a straight line was drawn to show the overall trend of the graph, then the drift can be thought of as the slope of that line, and the volatility describes the deviation from the slope (similar to standard deviation in probability).

In our application, we want to model changes in stock price \((S)\) with respect to time \((t)\). Consider \( Z = \ln S \). Then, according to the chain rule, \( dZ = \frac{dS}{S} \). Substituting these variables into the generalized Wiener process yields the following equation to describe the change in \( S \):

\[ dS = \mu S dt + \sigma S dW(t). \]  

(2)

This is helpful mathematically because as \( S \) gets close to zero, the "drift-like" quantity \( \mu S \) becomes very small. The same is true for the "volatility-like" quantity, \( \sigma S \). Also, this prevents \( S \) from becoming negative (since we assume \( S(0) > 0 \)), and non-negativity is good when dealing with financial markets. In our case, we assume that the drift, \( \mu \), is known and constant. Thus, once the drift is selected the first portion of our equation, \( \mu S dt \), is determined. This portion of the equation is known as the deterministic part of the equation. The second part of our equation has a Wiener Process in it, and is thus partly random. It is known as the stochastic portion of the equation. Another advantage of writing the equation in this manner is that once we add \( S \) to the right side of the equation, it goes from being a Wiener Process to an Itô Process, which allows us to apply Itô's Lemma in the following section. Unlike the generalized Wiener Process, the expressions on the right are not constant, but functions of time, \( t \), and the random variable, \( S \).

4 Itô's Lemma

4.1 What is Itô's Lemma?

Similar to the chain rule used in calculus, Itô's Lemma is a procedure for finding the differential of certain types of stochastic processes. Formally,
Itô’s Lemma states, “Suppose the random variable \( X \) is described by the Itô process
\[
dX = a(X, t)dt + b(X, t)dW(t)
\]
where \( dW(t) \) is a normal random variable. Suppose the random variable \( Y = F(X, t) \). Then \( Y \) is described by the following Itô Process.
\[
dY = \left( a(X, t)F_X + F_t + \frac{1}{2}(b(X, t))^2\right)F_{XX} dt + b(X, t)F_X dW(t)(\text{Buchanan 2008})
\]
(Note: Subscripts denote partial derivatives with respect to that variable. For example \( F_X \) is the partial derivative of \( F \) with respect to \( X \) and \( F_{XX} \) is the second partial derivative of \( F \) with respect to \( X \).) Instead of starting with \( dX \), though, we have \( dS \) as described in (2), and thus for our purposes we consider \( F(S, t) \) and not \( F(X, t) \). Similarly, instead of \( a(X, t)dt \) and \( b(X, t)dt \), we have constants in the form of \( \mu \) and \( \sigma \) in our equation and write
\[
dS = \mu dt + \sigma dW(t).
\]

4.2 Declaration of variables

Let \( S \) be the current value of the security. This is the price the stock is selling for in the market. Let \( t \) represent the time from today until the expiration date. The variable \( \mu \) is called the drift, and is considered constant. As stated in the previous section, it equals the overall slope of the graph for stock price. The volatility, \( \sigma \), is also considered to be constant and it represents the deviation from the drift, and is similar to standard deviation. The function \( W(t) \) is a Wiener Process, and is used to describe continuous random motion, which is used to model changes in stock price.

4.3 Proof of Itô’s Lemma

The general outline for this proof came from Buchanan’s book, but I filled in the gaps and focused it specifically on stock price. Start with the function \( F(S, t) \). Consider the general Taylor expansion
\[
dF = F_S dS + F_t dt + \frac{1}{2!}F_{SS}(dS)^2 + \frac{1}{2!}F_{St}dSdt + \frac{1}{2!}F_{tt}(dt)^2 + \cdots
\]
Now substitute \( dS = \mu dt + \sigma dW(t) \)
\[
dF = F_S(\mu dt + \sigma dW(t)) + F_t dt + \frac{1}{2!}F_{SS}(\mu dt + \sigma dW(t))^2
\]
Using stochastic calculus, one can show if $dt \to 0$, then $(dW(t))^2 = dt$, which comes from the idea that $E(W(t)^2) = dt$, an explanation of which is a bit beyond the scope of this paper. Thus, after applying the limit and ideas from stochastic calculus we can get to the common form of Itô’s Lemma

$$dF = (F_S\mu S + F_t + \frac{1}{2}F_SS\sigma^2S^2)dt + F_S\sigma SdW(t).$$

(3)

### 5 Derivation of the Black-Scholes PDE

Suppose the current value of a stock, $S$, obeys a stochastic process

$$dS = \mu Sdt + \sigma SdW(t)$$

as previously described in (2). If $F(S,t)$ is the value of any type of option (note that because Black-Scholes aims to find the correct price for an option by finding its value, value and cost will be used interchangeably to describe $F$), then according to Itô’s Lemma, $F$ obeys the stochastic process

$$dF = (\mu FS\mu S + \mu FS\sigma SdW(t) + \frac{1}{2}\sigma^2 F_S^2S^2 + 1 F_t)dt + \sigma FSdW(t).$$

(4)

Now suppose a portfolio of value $P$ is created by selling the option and buying $\Delta$ units of the security. Here, $\Delta$ does not mean “change in”, it is simply the number of units purchased. This is the standard notation used when dealing with the Black-Scholes PDE. The value of the portfolio is thus

$$P = F - \Delta S.$$  

(5)

Since the portfolio is a linear combination of the option and the security, the stochastic process governing the portfolio is

$$dP = d(F - \Delta S)$$

$$= dF - \Delta dS.$$  

(6)

Substituting equations (2) and (4) for $dF$ and $dS$ respectively to get

$$dP = (\mu FS\mu S + \frac{1}{2}\sigma^2 F_SS^2 + F_t)dt + \sigma FSdW(t)$$

$$-\Delta(\mu Sdt + \sigma SdW(t)).$$  

(7)
Rearranging gives us the following equations

\[
\begin{align*}
    dP &= (\mu S F_S - \Delta \mu S + \frac{1}{2} \sigma^2 S^2 F_{SS} + F_t)dt + F_S \sigma SdW(t) \\
    &\quad - \Delta \sigma SdW(t) \\
    &= (\mu S(F_S - \Delta) + \frac{1}{2} \sigma^2 S^2 F_{SS} + F_t)dt + (F_S - \Delta) \sigma SdW(t). \\
\end{align*}
\]  

(8)

Now to maximize the value of our portfolio, we see that \(\Delta = F_S\). Recall equation (5). Since we are trying to maximize the our portfolio, we can take the derivative of \(P\) (with respect to the stock price, \(S\)) and then set it equal to zero.

\[
\begin{align*}
    P &= F - \Delta S \\
    \frac{dP}{dS} &= \frac{dF}{dS} - \Delta \\
    0 &= \frac{dF}{dS} - \Delta \\
    \Delta &= F_S \\
\end{align*}
\]

How do we know this maximizes the equation instead of minimizes it? This is simple to check. Let \(\Delta < F_S\). Then \(\frac{dP}{dS}\) is greater than zero, and thus the value of the portfolio is increasing. If instead \(\Delta > F_S\), then \(\frac{dP}{dS}\) is less than zero and the value of the portfolio is decreasing. So at \(\Delta = F_S\) the value of the portfolio switches from increasing to decreasing and it is indeed the maximum. Since we are trying to maximize the value of our portfolio, assuming \(\Delta = F_S\) is reasonable. This helps us by reducing the number of stochastic terms. However, randomness is not eliminated because the value of the security \(S\) still remains and is stochastic. Substituting \(F_S - \Delta = 0\) into equation (8) gives

\[
    dP = \left(\frac{1}{2} \sigma^2 S^2 F_{SS} + F_t\right) dt. 
\]

(9)

Here, we recognize that in an arbitrage-free setting, the difference in returns from investing in the portfolio described above or investing an equal amount in a risk-free bond paying interest rate \(r\) should be zero. If arbitrage exists, it means that two financial instruments (in this case the portfolio and a bond) are mis-priced relative to each other, thus producing a betting strategy that always produces a positive net gain. We are assuming that
there exists an interest rate, $r$, such that the expected value on the return from a bond paying this interest rate is equal to the return from the portfolio described in equation (5). Thus the following equations are true.

$$0 = rP dt - dP$$
$$= rP dt - \left( \frac{1}{2} \sigma^2 S^2 F_{SS} + F_t \right) dt$$
$$= \frac{1}{2} \sigma^2 S^2 F_{SS} + F_t - rP$$ \hspace{1cm} (10)

In the last step we divided by $dt$ and rearranged the terms. Substitution of equation (5), $P = F - \Delta S$, gives

$$0 = \frac{1}{2} \sigma^2 S^2 F_{SS} + F_t - r(F - \Delta S)$$
$$= F_t + r\Delta S + \frac{1}{2} \sigma^2 S^2 F_{SS} - rF.$$

Finally, by substituting $\Delta = F_S$ we get the well-known Black-Scholes PDE,

$$0 = F_t + rSF_S + \frac{1}{2} \sigma^2 S^2 F_{SS} - rF.$$ \hspace{1cm} (11)

## 6 Solution to the PDE

The following solution imitates the one outlined by Buchanan in *An Undergraduate Introduction to Financial Mathematics*.

### 6.1 Recap

At this point in the process we have the PDE and three equations which describe the conditions for a European style call option. By rearranging equation (11) so that instead of being set equal to zero, it is set equal to $rF$, we arrive at the PDE

$$rF = F_t + \frac{1}{2} \sigma^2 F_{SS} + rSF_S.$$ \hspace{1cm} (12)

Time, $t$, can be anytime in between the current date ($t = 0$) and the strike time or expiry date, ($t = T$). Thus, this equation holds for $t \in (0, T)$. Since stock prices are strictly nonnegative, we get the condition $S \in (0, \infty)$. Now
considering only the strike price and the current value of the stock, the value of the option, $F$, is equal to the difference between the price at which the investor has the right to buy the stock, $K$, and the price at which the investor can then sell the stock, $S$. If the stock is worth less than the striking price, $K$, then the investor will not utilize his option, and it is worth $0$.

$$F(S, T) = (S - K)^+ \text{ for } S \in (0, \infty).$$  \hfill (13)

Thus, it follows that if the value of the stock ever reaches 0, the value of the option will also be 0.

$$F(0, T) = 0$$  \hfill (14)

The following equation comes from the Put / Call Parity Formula, which is discussed in Appendix A

$$F(S, t) \rightarrow S - Ke^{r(T-t)} \text{ as } S \rightarrow \infty \text{ for } t \in [0, T).$$  \hfill (15)

### 6.2 Change of variables

It is possible to change the variables in equation (12) so that we are left with the heat equation from physics, $u_\tau = u_{xx}$, which has a solution that was found by Joseph Fourier. Suppose we define $F$, $S$, and $t$ in terms of new variables $v$, $x$, and $\tau$ so that the following equations are satisfied:

$$S = Ke^{x} \iff x = \ln \left( \frac{S}{K} \right)$$  \hfill (16)

$$t = T - \frac{s\tau}{\sigma^2} \iff \tau = \frac{\sigma^2}{2}(T - t)$$  \hfill (17)

$$F(S, t) = K v(x, \tau)$$  \hfill (18)

We start by expressing $F_t$ in terms of $v$ and $\tau$. Taking the partial derivative of equation (18) with respect to $t$ gives

$$F_t = (Kv(x, \tau))_t.$$  \hfill (19)

Application of the multivariable form of the chain rule provides

$$F_t = K(v_x x_t + v_{\tau} \tau_t).$$  \hfill (20)

Here, $x_t = 0$ because $x = \ln(\frac{S}{K})$ is a constant; therefore

$$F_t = Kv_{\tau} \tau_t.$$  \hfill (21)
Next we consider the term \( \tau_t \). Since
\[
\tau = \frac{\sigma^2}{2} T - \frac{\sigma^2}{2} t
\]
we see that
\[
\tau_t = -\frac{\sigma^2}{2}. \tag{22}
\]
Substituting \( \tau_t \) into (21) yields
\[
F_t = \frac{-K\sigma^2}{2} v_t. \tag{23}
\]
Using similar methods, one can show
\[
F_S = e^{-x} v_x \tag{24}
\]
\[
F_{SS} = e^{-2x} (v_{xx} - v_x) \tag{25}
\]
and
\[
v_T = v_{xx} + (k - 1)v_x - k v \tag{26}
\]
where \( k = \frac{2x}{\sigma^2} \). Now we have a partial derivative of \( v \) with respect to \( T \) and three equations that describe conditions of \( v \). However, this equation does not yet resemble the heat equation since there are too many terms. Thus we introduce some constants that are chosen in such a way that terms can be eliminated. Let \( \alpha \) and \( \beta \) be constants. Then assume that a new dependent variable, \( u \), exists such that
\[
v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau) \tag{27}
\]
\[
v_x = e^{\alpha x + \beta \tau} (\alpha u(x, \tau) + u_x) \tag{28}
\]
\[
v_{xx} = e^{\alpha x + \beta \tau} (\alpha^2 u(x, \tau) + 2\alpha u_x + u_{xx}) \tag{29}
\]
\[
v_T = e^{\alpha x + \beta \tau} (\beta u(x, \tau) + u_T) \tag{30}
\]
\[
u_T = (\alpha^2 + (k - 1)\alpha - k - \beta) u + (2\alpha + k - 1)u_x + u_{xx}. \tag{31}
\]
Since \( \alpha \) and \( \beta \) are arbitrary constants, we can define them such that the coefficients of \( u_x \) and \( u \) are zero. Letting \( \alpha = \frac{1-k}{2} \) and \( \beta = \frac{-(k+1)^2}{4} \), we are left with the more widely known heat equation from physics:
\[
u_T = u_{xx}. \tag{32}
\]
Substituting the previous equations into our boundary conditions yields the following transformed boundary conditions:

\begin{align*}
  u(x, 0) &= \left(e^{(k+1)x/2} - e^{(k-1)x/2}\right)^+ \quad \text{for } x \in \mathbb{R} \quad (33) \\
  u(x, \tau) &\to 0 \quad \text{as } x \to -\infty \quad \text{for } \tau \in (0, T\sigma^2/2) \quad (34) \\
  u(x, \tau) &\to e^{k+1/2|x+(k+1)\tau/2|} - e^{k-1/2|x+(k-1)\tau/2|} \quad (35)
\end{align*}

as \( x \to \infty \) for \( \tau \in (0, T\sigma^2/2) \).

### 6.3 Introduction to Fourier Transforms

Fourier Transforms are used to transform certain types of PDEs (such as the heat equation) into ordinary differential equations (ODEs). The transform maps a function of one variable, say \( x \), to a function of a new variable, \( \omega \), usually thought of as a frequency. In our case, the Fourier Transform provides us with a very basic ODE which can be solved by separation of variables and integration of both sides. Let \( f(x) \) be Fourier Transformable\(^1\). Then, define \( \mathcal{F}\{x\} \) such that

\[
\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx.
\]

In other words, let \( \mathcal{F}\{x\} \) be the Fourier Transform of \( f \). Then

\[
\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(x)}{dx} e^{i\omega x} \, dx.
\]

Integrating by parts (\( \int udv = uv - \int vdu \)) with \( u = e^{i\omega x} \) and \( dv = \frac{df(x)}{dx} \, dx \) yields

\[
\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \left[ e^{i\omega x} f(x) \right]_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx
\]

\[
= \frac{e^{i\omega x}}{\sqrt{2\pi}} f(x) \Big|_{-\infty}^{\infty} - \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx.
\]

\(^1\) \( F \) is said to be Fourier Transformable if its domain consists of all real numbers, if \( F \) and \( F' \) are piecewise continuous on every interval of the form \([-M, M]\) for arbitrary \( M > 0 \), and if \( \int_{-\infty}^{\infty} |F(x)| \, dx \) converges (Buchanan 2008)
Here, we notice that in order for the Fourier Transform to exist, $f(x)$ vanishes as $x \to \pm \infty$ (Buchanan 2008). Thus, the leading integral is zero. Now,

$$\mathcal{F}\{f'(x)\} = -i\omega \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx \right].$$

Then, substitute (36) to get $\mathcal{F}\{f'(x)\} = -i\omega \mathcal{F}\{x\}$. We now use induction to generalize this to the $n^{th}$ degree. Above, we showed the base case for $n = 1$. Now, suppose it has been shown that $\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n \mathcal{F}\{x\}$. Then the following equalities are true:

$$\mathcal{F}\{f^{n+1}(x)\} = \int_{-\infty}^{\infty} f^{(n+1)}(x)e^{-i\omega x} dx$$

$$= f^{(n)}(x)e^{-i\omega x}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f^{(n)}(x)(-i\omega)e^{-i\omega x} dx.$$

This step used integration by parts. Next, we make use of the fact that in order for $F$ to be Fourier Transformable, the leading integral must be zero. Also, it is helpful to bring the constant terms, $i\omega$ outside of the integral.

$$\mathcal{F}\{f^{(n+1)}(x)\} = i\omega \int_{-\infty}^{\infty} f^{(n)}(x)e^{-i\omega x} dx$$

At this point it is apparent that within the integral is our base case: $\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n \mathcal{F}\{x\}$. Substitution provides the final two equalities:

$$\mathcal{F}\{f^{(n+1)}(x)\} = i\omega (i\omega)^n \mathcal{F}\{x\} = (i\omega)^{n+1} \mathcal{F}\{x\}$$

Thus, by the principle of mathematical induction, the result is true for all $n \in \mathbb{N}$.

### 6.4 Solution of PDE using Fourier Transform method

We start with our transformed PDE now in the form of the heat equation.

$$u_t = u_{xx}$$
for $x \in (-\infty, \infty)$, $\tau \in (0, \frac{\pi}{2})$. By applying the Fourier Transform to both sides of the equation we get

$$\mathcal{F}\{u_\tau\} = \mathcal{F}\{u_{xx}\}.$$  

By definition, this means

$$\int_{-\infty}^{\infty} u_\tau e^{-i\omega x} dx = \int_{-\infty}^{\infty} u_{xx} e^{-i\omega x} dx. \quad (37)$$

Looking at $u(x, \tau)$ and $f(x)$ and noticing $f(x)$ is constant with respect to $\tau$ we see

$$\frac{d}{d\tau} u(x, \tau) f(x) = u_\tau(x, \tau) f(x).$$

And thus

$$\int \frac{d}{d\tau} u(x, \tau) f(x) dx = \frac{d}{d\tau} \int u(x, \tau) f(x) dx$$

because the $\frac{d}{d\tau}$ is not affected by taking an integral with respect to $x$. Hence, we can rewrite the left side of equation (37) as

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} u e^{-i\omega x} dx.$$  

Now we deal with the right side of the equation. Using the property discussed in the previous section,

$$\int_{-\infty}^{\infty} u_{xx} e^{-i\omega x} dx = -\omega^2 \int_{-\infty}^{\infty} u e^{-i\omega x} dx$$

where the negative comes from $i^2$ and the square comes from the order of the partial derivative of $u$. We have thus rewritten equation (37) as

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} u e^{-i\omega x} dx = -\omega^2 \int_{-\infty}^{\infty} u e^{-i\omega x} dx.$$  

The identical part of this equation, $\int_{-\infty}^{\infty} u e^{-i\omega x} dx$, is the definition of the Fourier Transform of $u$, which we denote $\hat{u}$. Thus, we are left with an ordinary differential equation,

$$\frac{d\hat{u}}{d\tau} = -\omega^2 \hat{u}. \quad (38)$$
To solve this equation take the following steps: first, separate the variables.

\[ \frac{1}{u} d\hat{u} = -\omega^2 d\tau \]

Then integrate both sides of the equation.

\[ \int \frac{1}{u} d\hat{u} = \int -\omega^2 d\tau \]

\[ \ln |\hat{u}| = -\omega^2 \tau + C \]

Finally, isolate \( \hat{u} \) to get

\[ \hat{u} = De^{-\omega^2 \tau} \]

where \( D \) is constant with respect to \( \tau \). Next, to find \( D \), we use our initial conditions. Set \( \tau = 0 \), that is, \( \hat{u}(\omega, 0) = D \). Thus, \( D \) is the Fourier Transform of the initial equation

\[ \hat{u}(x, 0) = (e^{(k+1)x/2} - e^{(k-1)x/2})^+. \]

For simplicity, we write \( D = \hat{f}(\omega) \), which gives

\[ \hat{u}(\omega, \tau) = \hat{f}(\omega)e^{-\omega^2 \tau}. \]  \hspace{1cm} (39)

The next step toward our final solution is to start moving back toward the original variables from the Black-Scholes PDE. We start by taking the inverse Fourier transform,

\[ F^{-1}\{\hat{u}(\omega, \tau)\} = F^{-1}\{\hat{f}(\omega)e^{-\omega^2 \tau}\}. \]

This yields

\[ u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} (e^{(k+1)z/2} - e^{(k-1)z/2})e^{-x^2/(4\tau)}dz. \]

Next, we substitute \( z = x + \sqrt{2\tau}y \), remembering to also substitute for \( dz \),

\[ u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{(k+1)(x+\sqrt{2\tau}y)/2} - e^{(k-1)(x+\sqrt{2\tau}y)/2})e^{-y^2/2}dy. \]  \hspace{1cm} (40)

To simplify this equation, we note that
\[ e^{(k+1)(x+\sqrt{2}\pi y)/2} - e^{(k-1)(x+\sqrt{2}\pi y)/2} > 0 \]
\[ \Leftrightarrow e^{(k+1)(x+\sqrt{2}\pi y)/2} > e^{(k-1)(x+\sqrt{2}\pi y)/2}. \]

Looking at just the exponents, we can find an expression in terms of \( y \) that we can use to change the lower limit of integration and drop the + sign within the integral.

\[
k(x + \sqrt{2}\pi y) + (x + \sqrt{2}\pi y) > k(x + \sqrt{2}\pi y) - (x + \sqrt{2}\pi y)
\]

\[
x + \sqrt{2}\pi y > -(x + \sqrt{2}\pi y)
\]

\[ \Leftrightarrow x + \sqrt{2}\pi y > 0 \]

\[ \Leftrightarrow y > \frac{-x}{\sqrt{2}\pi} \]

Thus, we can change the lower limit and distribute, and equation (40) becomes

\[
\frac{1}{\sqrt{2}\pi} \int_{-x/\sqrt{2}\pi}^{\infty} e^{[(k+1)(x+\sqrt{2}\pi y)-y^2]/2} - e^{[(k-1)(x+\sqrt{2}\pi y)-y^2]/2} dy.
\]

The next step is to complete the square in the exponents. We do this for each term. This gives us

\[
u(x, \tau) = \frac{1}{\sqrt{2}\pi} \int_{-x/\sqrt{2}\pi}^{\infty} e^{-y(\frac{(k+1)^2}{2})/2 + \frac{(k+1)2\pi + (k+1)x}{2} + (k+1)/2} dy
\]

\[ - \frac{1}{\sqrt{2}\pi} \int_{-x/\sqrt{2}\pi}^{\infty} e^{-y(\frac{(k-1)^2}{2})/2 + \frac{(k-1)2\pi + (k-1)x}{2} + (k-1)/2} dy
\]

\[ = \frac{e^{(k+1)^2/4}e^{(k+1)x/2}}{\sqrt{2}\pi} \int_{-x/\sqrt{2}\pi}^{\infty} e^{-y(\frac{(k+1)^2}{2})/2} dy
\]

\[ - e^{(k-1)^2/4}e^{(k-1)x/2} \int_{-x/\sqrt{2}\pi}^{\infty} e^{-y(\frac{(k-1)^2}{2})/2} dy.
\]

Instead of computing this integral, we transform it until it matches something we recognize. If we substitute \( w = y - \frac{1}{2}(k+1)\sqrt{2}\pi \) into the first integral
and recognize $dw = dy$, then (ignoring the coefficient) we have

$$\int_{-\infty}^{\infty} x/\sqrt{2\tau} + \frac{1}{2}(k+1)/\sqrt{2\tau} e^{-w^2/2} dw. \tag{42}$$

To change the limits of integration, consider $y = -x/\sqrt{2\tau}$; then

$$w = -x/\sqrt{2\tau} - \frac{1}{2}(k - 1)/\sqrt{2\tau}
= - \left[ \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)/\sqrt{2\tau} \right].$$

We also made use of the identity $\int_{-y}^{b} = -\int_{b}^{a}$. Next, we recognize equation (42) as the cdf for a standard normal distribution. Thus, using $\Phi$ to denote the cdf, this integral can be written as

$$\sqrt{2\pi} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)/\sqrt{2\tau} \right).$$

For the second integral, we will substitute $w* = y - \frac{1}{2}(k - 1)/\sqrt{2\tau}$. This yields

$$\sqrt{2\pi} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)/\sqrt{2\tau} \right).$$

Combining these two equations with our coefficients we have the equation

$$u(x, \tau) = e^{(k+1)^2/2 + (k+1)^2\tau/2} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)/\sqrt{2\tau} \right)
- e^{(k-1)x/2 + (k-1)^2\tau/4} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)/\sqrt{2\tau} \right)$$

At this point we change back to our original variables. By rearranging equation (27) and using the values for $\alpha$ and $\beta$ that gave us the heat equation, we find

$$v(x, \tau) = e^{\frac{1}{2}(1-k)x - \frac{1}{4}(k+1)^2\tau}u(x, \tau).$$

Canceling terms in the exponents, we are left with

$$v(x, \tau) = e^x \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k + 1)/\sqrt{2\tau} \right) - e^{-k\tau} \Phi \left( \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k - 1)/\sqrt{2\tau} \right). \tag{43}$$

This $w = y - \frac{1}{2}(k + 1)/\sqrt{2\tau}$ is different than $\omega$ introduced earlier when dealing with Fourier Transforms. Later, we will substitute again for $w$ to make our equation more concise.

The cdf (or cumulative distribution function) of a standard normal distribution is defined as $P(X < x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$. (Ross 2006)
Next, we use equations (16–18) to transform back into our original variables, so that we arrive at the Black-Scholes European Call Option Pricing Formula,

\[ v(x, \tau) = \frac{S}{K} \Phi(w) - e^{-r(\tau-t)} \Phi(w - \sigma\sqrt{T-t}). \]  

(44)

Then by equation (18) we have

\[ F(S, t) = S \Phi(w) - Ke^{-r(T-t)} \Phi(w - \sigma\sqrt{T-t}). \]

For meaningful notation, we write \( C \) instead of \( F \) to denote that we are dealing with a call option and not a put option, giving us the Black-Scholes European Call Option Pricing Formula,

\[ C(S, t) = S \Phi(w) - Ke^{-r(T-t)} \Phi(w - \sigma\sqrt{T-t}). \]  

(45)

7 Application and Limitations of Model

7.1 Application

In order to consider how the Black-Scholes equation is applied we will consider an example. Suppose Joe Investor walks into the Chicago Board Option Exchange looking to buy a call option in Fictional, Inc. Given the following circumstances, what should the price of a European style call option be according to the Black-Scholes model? The stock is selling for $62, Joe wants a striking price of $60 and an expiry date 5 months from now. The continuously compounded interest rate is 10% per year, and the volatility of the price of the stock is assumed to be 20% per year. Thus, we have \( T = 5/12 \) because time is always measured in years, \( r = 0.10, \sigma = 0.20, S = 62, \) and \( K = 60. \) First, we must find the value of \( w, \)

\[ w = \frac{\ln(\frac{62}{60}) + (.10 + .20^2/2)(5/12 - 0)}{.20\sqrt{5/12 - 0}} \approx .641287 \]

and then we can plug in all of our other values into the Pricing Formula, which gives

\[ F(62, 0) = (62)\Phi(.641287) - (60)e^{-1(5/12-0)}\Phi(.641287(.2)\sqrt{5/12 - 0}) \approx 5.80. \]

This tells us that the call option should be priced at $5.80 per option.
7.2 Limitations of the model

Of course, all models have their limitations because assumptions must be made. We will now consider how the assumptions we made during the derivation affect the effectiveness of the Black-Scholes model. First, the model assumes the stock price is continuous (meaning it assumes large, abrupt changes do not occur). However, when large mergers are announced, there are often huge unpredictable changes in stock price over a short period of time. The model also assumes that the volatility is constant. However, the volatility fluctuates based on the conditions of the market. It, unlike other inputs for the model, can only be estimated. Overall, the model underestimates the likelihood of large changes in the price of the stock.

Another assumption is that no dividends are paid until after expiration. While dividends are only paid to shareholders, and not to owners of options, dividends do affect the value of the stock. One way to correct for this inconsistency is to subtract the value of the future dividend from the stock price, \( S \) within the formula.

This model also only applies to European style options, which can only be exercised at the expiry date. Although, these are easier to deal with mathematically, in the financial world American style options are much more popular as they give the owner of the option more freedom and potential for profit. If the stock price rises, the owner of the option does not need to wait until the date of expiration in hopes the price of the stock stays high; instead, he or she can exercise his or her right at any time until the expiration date.

The Black-Scholes model assumes there exists some known risk-free interest rate at which the investment is continuously compounded. This interest rate does not exist, but is usually substituted for the discount rate on United States Government Treasury Bills with 30 days until maturity.

8 Conclusion

As stated above, overall the Black-Scholes model tends to underestimate the probability of great changes in the stock market because it assumes a known and constant volatility. Thus, it is fairly accurate for short term options, but tends to lose its dependability for mid to long term options. However, due to its simplicity (most of the values for input are easily attained, estimated, or assumed like the Treasury Bill) it is still very popular for the pricing
of options in the financial markets. At the very least, it provides a good estimate and serves as a good jumping off point for determining an option's value.
Appendices

A Put-Call Parity Formula

The Put-Call Parity Formula deals strictly with European style options and states the following:

\[ P + S = C + Ke^{-rT} \]  \hspace{1cm} (46)

The \( P \) in this case is the value or price of a European put option, and similarly \( C \) is the value of a European call option. \( S \) represents the stock price as it has throughout this paper. \( K \) once again is the striking price, \( T \) is the expiration date represented as a fraction of years (e.g. \( 5/12 \) for 5 months) from the current date, and \( r \) represents the interest rate at which an investor can borrow. To see where this equation comes from, consider the left side of the equation to be one portfolio (say Portfolio A) and the right side to be another portfolio (Portfolio B). The Put-Call Parity Formula states that in an arbitrage-free setting these portfolios must be equal to each other. To prove this, use two separate proofs by contradiction. First, assume that Portfolio A is worth less than Portfolio B, i.e.

\[ P + S < C + Ke^{-rT} \]

If an investor borrows an amount equal to \( P + S - C \) at interest rate \( r \), then this allows the investor to buy the Put option and sell the Call option. At the strike time (\( t = T \)), the investor must then pay back the principal amount and interest in the amount of \((P + S - C)e^{rT}\). If \( S > K \) at time \( T \) then the Put will be worthless and allowed to expire and the investor will exercise the Call option. The security will be sold at \( K \). Thus the net proceeds are \( K - (P + S - C)e^{rT} > 0 \) since this matches the inequality listed above. If instead \( S < K \) at time \( T \) then the Call will be allowed to expire and the Put will be exercised by the investor. In this case the proceeds are the same as the previous case \((K - (P + S - C)e^{rT})\) and there exists a risk-free profit if Portfolio A is worth less than Portfolio B. Thus, in an arbitrage-free setting our initial assumption cannot be true.

Conversely, let us assume this time that Portfolio A is worth more than Portfolio B. That is

\[ P + S > C + Ke^{-rT} \]
It is possible for an investor to sell the stock and the Put option, and to buy the Call option. Thus, there is an initial positive flow of capital in the amount of $S + P - C$. Now this investor can invest this amount in a risk-free bond earning some interest rate $r$. At time $T$ the investor will have $(S + P - C)e^{rT}$. If $S > K$ at this time, the investor will exercise the Call and not the Put as the Put will be worthless. This leaves the investor with a net gain of $(P + S - C)e^{rT} - K > 0$ since this inequality is equal to the one listed above. If $S < K$ at time $T$, then the Call expires unused and the investor who owns the Put option will exercise it. Thus the investor will buy the stock at price $K$ and their net gain is the same as before. Hence, if Portfolio B is worth less than Portfolio A, an arbitrage opportunity exists. Therefore, the two portfolios must have the same value and the Put-Call Parity Formula must be true.

In terms of the boundary condition described in the Solution section of the paper, consider the case as $S \to \infty$. A Put option becomes worthless as the stock price becomes increasingly big because it is not beneficial to the investor to pay for the right to sell a stock at price much lower than market value. Thus, as $S \to \infty$, $P \to 0$, and the Put-Call Parity Formula becomes $C \geq S - Ke^{-rT}$. Also as $S \to \infty$, the difference $S - K \approx S$. Thus, we have the boundary condition, $C \to S - Ke^{r(T-t)}$ for $t \in [0, T)$. 


Bibliography

