Why Didn't Euclid Think of That?: The History and Development of Geometry in the Western World

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University of Redlands

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UNIVERSITY OF REDLANDS

WHY DIDN'T EUCLID THINK OF THAT?:
THE HISTORY AND DEVELOPMENT
OF GEOMETRY IN THE WESTERN WORLD

A Paper Prepared for the Senior Project
in Partial Fulfillment of
the Requirements for the Degree
Bachelor of Science in Mathematics

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ABSTRACT

The subject of geometry has been in existence around the world for thousands of years and is a topic that continues to be taught in high school classrooms around the world today. When one thinks about geometry, the name Euclid and concepts such as proving triangles are congruent and examining properties of parallel lines usually come to mind. In fact, most of what is taught in the education system today can be found in Euclid's *Elements*, a compilation of what was known about geometry up until around 300 B.C.E. But what has become of geometry since Euclid's time and why was there a gap in its production in Western cultures that lasted for hundreds of years? This project investigates the origins of geometry in the Western world and its significance throughout history. In particular, it examines several important theorems and their proofs that were not discovered until over 1500 years after Euclid's time, and yet the proofs are based on ideas found in the *Elements*. This paper not only shows how geometry has developed up until this present time but also gives the reader a brief account of how geometry is expanding today and where it is heading in the future.
1. ORIGINS OF GEOMETRY

1.1 The Rising Need

Like any other topic, idea, or invention, the field of geometry probably did not exist until the need for it arose. One of the circumstances leading to a need for geometry was in the New Stone Age, when the nomadic way of life in which hunting and gathering were prominent was replaced by a more agricultural lifestyle. Not only was there a change in living but early humans began to question why the objects in the heavens moved as they did and wondered what caused the seasons to change. All of these questions “contributed to the blend of early science, mathematics and astronomy with early religion, astrology and myths” (Holme, 2002, p. 4). Not only were connections made between mathematics and other sciences, but geometric patterns have been found on pottery and textile fragments from this time which today are studied in the field of Ethnomathematics.

1.2 The Great River Civilizations

In the Great River Civilizations such as those of the Mesopotamians and the Egyptians, urban centers were built in which running water was supplied for irrigation for farms along with a drainage system. Perhaps one of the most important civilizations in terms of geometric progression was the Indus civilization. In the earliest Indian mathematical texts known, which were written around the sixth century B.C.E., “there are specific geometric rules for constructing altars” out of strings or rope (Holme, 2002,
p. 11). Known as the Sulbasutras, which means “string rules”, these three books also conveyed knowledge of the Pythagorean Theorem and the concept of similar right triangles. Also linked to the beginnings of geometry is the need for a calendar to keep track of the changing of seasons for harvesting, which combines mathematics with the field of astronomy. Land measurement and the construction of irrigation channels make use of important concepts of geometry as well. When looking at all the ways in which geometry was applied to these civilizations’ way of life, it is unique to this field that “the earliest mathematics we encounter is qualitatively of the same nature as the mathematics of today” (Holme, 2002, p. 11).

Today, the Egyptian Civilization is remembered most for its great pyramids. Most people, however, forget that the pyramid is a common geometric object. Although the three dimensional pyramids that still stand in the land of Egypt are considered today as one of the ancient wonders of the world, most people do not know about another great pyramid found on an ancient piece of papyrus known as the Moscow Papyrus (Holme, 2002). These papers date back to 1850 B.C.E. and contain 25 mathematical problems of common geometric equations and examples such as the volume of a frustum of a square pyramid and the surface area of a hemisphere. In the formula for the surface area of a hemisphere, the Egyptians estimated the value of pi to be $3\frac{1}{6}$, although many researchers argue about the exact interpretation of the material found on the papyrus.
1.3 Greek Geometry

1.3.1 Thales and Pythagoras

Among the most well known contributors to geometry are the Greeks, who introduced the concept of mathematical proof. In fact, the word geometry is derived from the Greek word \( \gamma\varepsilon\omega\mu\varepsilon\tau\pi\alpha \) which translates to “earth measure”. The first Greek mathematician whose name is known is Thales of Miletus who lived from 625-545 B.C.E. Thales is said to be known as the founding Father of Greek Geometry and is credited with finding the height of pyramids by measuring their shadow at the exact time the shadow cast from himself was the same length as his height (Holme, 2002). He is said to have discovered that the base angles of an isosceles triangle are congruent, vertical angles are congruent, and a diameter of a circle cuts the circle into two equal parts.

Another common name associated with the field of geometry is that of Pythagoras of Samos who was born around 570 B.C.E. Pythagoras’ time in Samos was brief, however, as he left because some said that “his teachings [were] too abstract and symbolic” (Holme, 2002, p. 35). Pythagoras was a very important leader of his time who founded his own brotherhood known as The Pythagoreans in Croton, a city on the coast of Southern Italy. One of the doctrines of the Pythagoreans is that “the natural numbers formed the basic organizing principle for everything” (Holme, 2002, p. 38). They also used the idea of a ratio in many aspects of their lives such as the motion of the planets and musical harmonies.
As for the geometry known by the Pythagoreans, the list is quite lengthy. Although some of the Pythagorean’s discoveries were definitely known before their time, they are most often credited with the following:

1. The Pythagoreans knew that the sum of the angles of a triangle is equal to two right angles.
2. The Pythagorean Theorem: The Pythagoreans knew that in a right triangle the square on the hypotenuse is equal to the sum of the squares on the two sides containing the right angle.
3. The Pythagoreans knew several types of constructions by straight-edge and compass of figures of a given area (Holme, 2002, p. 38-39).

Another important aspect of Greek geometry is the discussion of the three classical problems. The following problems were worked on for centuries and have been proven to be insoluble using only a straightedge and compass.

1. Squaring the Circle: Given any circle, it is impossible to construct a square with the same area as the one enclosed by the circle.
2. Doubling the Cube: Given any cube, it is impossible to construct the side of another cube whose volume is twice that of the given cube.
3. Trisecting the Angle: Given any angle, it is impossible to divide it into three equal parts.
1.3.2 Euclid and His Elements

Although the Greek geometers mentioned thus far made significant contributions to the field, one of the most well known Greek mathematicians is Euclid of Alexandria. Euclid lived around 300 B.C.E. and probably “received his training at the Academy from the followers of Plato” (Dunham, 1997, p. 30). His greatest contribution to geometry and arguably one of the greatest contributions of all time to the field of mathematics is his book known as the *Elements*. In this book which is divided into 13 smaller books, “Euclid collected and systematized the entire body of mathematics known to his time” (Holme, 2002, p. 67).

Euclid based his writing on a fundamental idea known as The Hypothetical-Deductive Method. This method states that “all known geometric facts or theorems should be deduced by agreed upon logical rules of reasoning from a set of initial, self evident truths, called postulates” (Holme, 2002, p. 68). In other words, these postulates should not be questioned as to whether they were true or not. Furthermore, the postulates had to be kept to as small a number as possible, only being used when it is impossible to deduce anything from an already given concept.

It must be noted that “Euclid’s great genius was not so much in creating a new mathematics as in presenting the old mathematics in a thoroughly clear, organized, and logical fashion.” Euclid organized the field of geometry into an axiomatic system that avoided “circularity in reasoning” and was consistent, meaning that it was free from contradiction (Dunham, 1997, p. 31). Unfortunately, the *Elements* are by no means an easy read. When asked by King Ptolemy I if there was any easier way to learn geometry,
Euclid replied “no, to the geometry there is no separate road for kings, there is no Royal Road to Geometry” (Holme, 2002, p. 68).

One of the most important postulates in the Elements is Euclid’s Fifth Postulate. This postulate states that “if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.”

Euclid’s Fifth Postulate is one of the most important postulates in the Elements for two main reasons. First of all, no one was able to prove that it was independent of the other postulates for about 2000 years. Secondly, this postulate allows for what we call Euclidean Geometry, claiming that given a line and a point not on the line in a plane, there is one and only one line through the point parallel to the given line. Slightly altering this postulate creates two non-Euclidean geometries known as Spherical geometry and Hyperbolic geometry in which the standard concept of parallelism no longer exists.
2. PERIOD OF LIMITED GEOMETRIC PROGRESSION IN GREECE AND THE WEST

2.1 Importance of Verbal Interpretation

An important point to mention about Greek geometry as a whole is that it thrived on being interpreted verbally by the great mathematicians of the time. It was vital to not only study geometry in written form because the absence of "the verbal form of exposition made it very difficult to resume work in geometry once the line of transition from person to person had been interrupted" (Holme, 2002, p. 74). Therefore, once all the great geometers had been dead for a hundred years, it was very difficult to continue the work of geometry only from written sources.

2.2 Destruction of Geometric Records

The death of the great Greek geometers was not the only reason why there was a huge decrease of new material being discovered in the field of geometry in Western cultures. Although there were several mathematicians deserving mention after this time such as Heron and Menelaus, the focus of most people was on the many wars raging in the Western world. These wars were responsible for the destruction of many of the written records of geometric ideas. For example, during a war in Egypt, Caesar of Rome's communication at sea was being threatened by his enemy so he put fire to the enemy's ships, successfully destroying them. Unfortunately, the fire spread to land and the great Library of Alexandria, which held most of the written development of geometry
up to that time, was destroyed. Around 30 B.C.E. the library was resupplied with books only to be destroyed again later by Christians who burned all material in Alexandria that they considered as pagan.

2.3 Period of Regeneration

Since most of the books containing the material in Euclid's *Elements* were destroyed, the field of geometry in Western civilizations was now only going to be preserved by writing more material. Pappus of Alexandria, who lived from about 290-350 C.E., headed a school in Alexandria which helped the rewriting process. His major work entitled *Synagoge* or the *Mathematical Collection*, was a collection of eight books that was "written with the object of reviving the classical Greek geometry [and] covers practically the whole field." Although it was only intended to serve as a guide to classical geometry, at the time it was a "worthy representative" (Holme, 2002, p. 120).

In 630 C.E., the Arabs conquered Alexandria. They also contributed to saving ancient mathematics by keeping in their possession many fundamental texts, including the *Elements*, that were later translated from Arabic into Latin. The Arabs took the ruined city of Alexandria and founded academies for mathematics to begin the rebuilding process. In Europe, the fall of the Roman Empire brought about the Dark Ages, which lasted until about the middle of the 11th century, and during this time almost no mathematical activity in the Western world took place. In the 12th century, books that were translated from Arabic to Latin finally began circulating in Europe. This was happening primarily in Byzantium, southern Spain, and Italy due to the contact between
European Moslems, whose scientific language was Arabic, and the European Christians and Jews, whose scientific languages were Latin and Hebrew, respectively.
3. CHANGES IN GEOMETRIC METHODS

3.1 Modern Geometric Tools Employed

During this rebuilding of the subject of geometry, many new tools used to solve problems in geometry also came into existence. For example, instead of figures being built with only lines, points, and circles, more modern geometry has expanded its tools to include general conic sections, curves of higher degrees, and even transcendental curves.

These new tools have become very important in solving important problems from ancient times that were considered insoluble. Recall the three classical problems from Greek mathematics in which no solution could be found using only a straightedge and compass. Using the new methods mentioned above, all three problems can actually be solved (Holme, 2002). Geometry involved the use of synthetic methods up until the 15th century. Then these old methods were expanded by new techniques by great mathematicians such as Desargues, Pascal, and Descartes. Gérard Desargues was born in 1591 and lived in Lyon, France. He became an engineer and later moved to Paris where he spent most of his time studying geometry. Perhaps his most famous work was a treatise on conic sections which he
published in 1639 (Holme, 2002). It was lost however until 1951, when a copy resurfaced that contained a proof of Desargues' now well-known Theorem of Perspective. Blaise Pascal, born in 1623, also contributed to these newer methods of geometry. Like Desargues, he wrote about conic sections and contributed heavily to projective geometry.

3.2 Analytical Geometry

Although there were many great mathematicians rapidly progressing the field of geometry at this time, it was the work of René Descartes that took geometry to a whole new level. In 1637, he published a book on philosophy called Discours de la méthode. It was an appendix in the book that caught the interest of mathematicians however. The appendix, named La géométrie, "provided the first published account of what we now call analytic geometry". Although the ideas in the appendix were fairly obvious and not very complex, they "announced the marriage of algebra and geometry that would become indispensable in all subsequent mathematical work" (Dunham, 1997, p. 157). A few of his ideas contained in his appendix are as follows:

1. The set of real numbers are identified with the points on a line.
2. A point in a plane is represented by a pair \((x, y)\) of real numbers.
3. \(x\) and \(y\) are called the coordinates of the point \(P = (x, y)\).
4. A line is defined as \(ax + by + c = 0\), where \(a\), \(b\), and \(c\) are fixed real numbers.

In other words, Descartes "reintroduced algebra into geometry" which was done by the Babylonians as well thousands of years ago (Holme, 2002, p. 137).
The mathematics brought about in the 18th century also contributed to the development of geometry. Colin Maclaurin who is best known for his work on series expansion in the field of calculus also worked on problems in geometry. His most important contribution lies in his "ingenious construction for trisecting an angle in three equal parts by means of a curve of degree three, known today as the Trisectrix of Maclaurin" (Holme, 2002, p. 140).
4. MATHEMATICAL PROOFS FROM CLASSICAL GEOMETRY

4.1 Euler's Triangle Inequality

Another famous mathematician of this time was Leonhard Euler who was born in Switzerland in 1707. Euler was a prolific writer whose work accounted for one third of all mathematical written publications in the 18th century (Dunham, 1997). Although he contributed to almost every branch in mathematics of his time, we will focus on one of his additions to the area of geometry, namely his triangle inequality. The theorems and definitions below are needed to understand the following proof:

**Theorem:** All three angle bisectors of a triangle intersect in a single point called the incenter.

**Theorem:** There is a circle, called the incircle, which is centered at the incenter and is tangent to all three sides of a triangle.

**Definition:** A radius of an incircle is called an *inradius*.

**Theorem:** There is a unique circle containing any three noncollinear points in a Euclidean plane.

**Definition:** The circumcircle of a triangle is the unique circle that contains all three of its vertices.

**Definition:** A radius of a circumcircle is called a *circumradius*.

**Definition:** A convex quadrilateral \( ABCCD \) is a Saccheri Quadrilateral if \( m\angle A = m\angle B = 90^\circ \) and \( AD = BC \).
Euler's Triangle Inequality: In any triangle, the circumradius $R$ and the inradius $r$ satisfy $R \geq 2r$, with equality if and only if the triangle is equilateral.

In order to prove Euler's Triangle Inequality, we will use the arithmetic mean-geometric mean inequality together with three lemmas.

![Figure 2: Relating the Circumradius and the Inradius](image)

**Arithmetic Mean-Geometric Mean Inequality:** For any two positive numbers, the arithmetic mean is at least as great as the geometric mean. In other words, if $x$ and $y$ are both positive, \( \frac{x+y}{2} \geq \sqrt{xy} \), where the two are equal if and only if $x = y$.

Applying this formula to all possible pairs of three numbers $x$, $y$, and $z$ yields the following result: $x + y \geq 2\sqrt{xy}$, $y + z \geq 2\sqrt{yz}$, $z + x \geq 2\sqrt{zx}$. If we multiply these three inequalities together we get $(x+y)(y+z)(z+x) \geq 8xyz$.
Consider $\Delta ABC$ with side lengths $a$, $b$, and $c$ and construct the incircle of the triangle. To find the incenter of this circle, construct the angle bisectors of each vertex of the triangle and find the point of concurrency. We know by a theorem (Kay, 2001, p.182) that since the incenter lies on all three angle bisectors of our triangle, it is equidistant from the sides of the three angles, and hence, equidistant from all three sides of the triangle. Therefore, if we drop a perpendicular line from the incenter to each side of the triangle, these three lines are of equal length, $r$, and furthermore are inradii of the incircle. We now have six right triangles with the following side lengths and angle measures in the figure below (Figure 3 based on Nelsen):

![Figure 3: Constructing the Inradius](image)

As seen from the figure, $x + y = c$, $y + z = a$, and $z + x = b$, which substituting these values into the inequality yields $abc \geq 8xyz$. We will now prove three lemmas that when applied to $abc \geq 8xyz$ lead to the proof of Euler's Triangle Inequality.

**Lemma 1:** $4KR = abc$, where $K$ is the area of $\Delta ABC$ and $R$ is the circumradius.

(Figure 4 based on Nelsen):
Figure 4: The Circumcircle of \( \triangle ABC \)

We know that in \( \triangle ABC \), at least two of the angles must be acute since all three angle measures must add up to 180°. Assume without loss of generality that \( \angle A \) is an acute angle. Begin by constructing the circumcircle of \( \triangle ABC \) with side lengths \( a, b, \) and \( c \). Construct altitude \( CD \) where \( D \) lies on \( AB \) and construct the circumcenter \( O \) of the circumcircle. Let \( h = CD \) and \( F \) be the midpoint of \( CB \). We know by the perpendicular bisector theorem and Axiom I-1 (Kay, 2001), that since \( F \) is equidistant from points \( C \) and \( B \) by definition of midpoint and \( O \) is equidistant from points \( C \) and \( B \) since \( OC \) and \( OB \) are circumradii of the circumcircle, that points \( O \) and \( F \) lie on the perpendicular bisector of \( CB \) and hence \( OF \perp CB \).

We now claim that \( m\angle CAD = m\angle COF (= \theta) \). This is true because \( \triangle COF \cong \triangle BOF \) by the Hypotenuse Leg Theorem (Kay, 2001). Then, by CPCF

\( \angle COF \cong \angle BOF \). Since \( F \) is the midpoint of \( CB \), we know that \( CF = FB = \frac{a}{2} \). Let \( X \) be a point on the circumcircle on the opposite side of \( BC \) as \( A \). By definition of arc
measure $m\angle CXB = m\angle COB$ and by the Inscribed Angle Theorem (Kay, 2001)

$m\angle CAB = \frac{1}{2} m\angle CXB \Rightarrow m\angle CXB = 2m\angle CAB$. Since both of these equations are equal to $m\angle CXB$ we equate them to get $m\angle COB = 2m\angle CAB$ and taking $\frac{1}{2} m\angle COB = m\angle COF$ yields the desired result that $m\angle CAD = m\angle COF$.

By the AA Similarity Criterion, $\triangle ACD \sim \triangle OCF$ since $m\angle CAD = m\angle COF(= \theta)$ and $m\angle ADC = m\angle CFO = 90^\circ$. Applying the definition of similar triangles,

$$\frac{h}{b} = \frac{2}{R} \Rightarrow h = \frac{ab}{2R}. \text{ Since } K = \frac{1}{2} ch, \text{ we get}$$

$$K = \frac{1}{2} ch \Rightarrow K = \frac{1}{2} c \left( \frac{ab}{2R} \right) \Rightarrow K = \frac{abc}{4R} \Rightarrow 4KR = abc. \quad \blacksquare$$

**Lemma 2:** $x+y+z = r^2(x+y+z)$

We first work to construct a rectangle that is composed of triangles similar to those in figure 3 (Figure 5 based on Nelsen):

![Figure 5: Creating a Rectangle from Similar Triangles](image-url)
To construct such a rectangle, multiply all the side lengths of triangle 1 by \( rz \) and let \( w = \sqrt{r^2 + x^2} \). Then, multiply all the side lengths of triangle 2 by \( r(x+y) \) and all the side lengths of triangle 3 by \( wz \). Finally, again take a copy of triangle 1 and multiply all the side lengths by \( yz \). Now rearrange the four triangles listed above to create \( \triangle ACEF \) as shown in the figure above.

Now it must be proven that \( \triangle ACEF \) is indeed a rectangle. In doing so, we will show that points C, D, and E are collinear and that \( \overrightarrow{FD} \) in triangle 3 and \( \overrightarrow{FD} \) in triangle 2 are really the same segment. We begin by showing that the angle sum at vertex B is equal to \( 180^\circ \), and therefore \( \overrightarrow{AC} \) is a straight line. In similar triangles, corresponding angles are congruent. Hence, in figure 5, \( m \angle ABF = a \) and \( m \angle CBD = a \). From triangle 1 in the figure above and to the left, we know that \( a + a + 90^\circ = 180^\circ \) since the angle sum of any triangle is \( 180^\circ \). Therefore since \( m \angle B = a + 90 + a \) by the Angle Addition Postulate, we can deduce that the angle sum at B equals \( 180^\circ \). Now consider the angle sum at vertex F. We can show that \( m \angle F = \alpha + \beta + \lambda = 90^\circ \), and therefore, \( \angle F \) is a right angle. To see why this is true, note that in the left figure above, \( 2\alpha + 2\beta + 2\lambda = 180^\circ \) which implies that \( \alpha + \beta + \lambda = 90^\circ \).

We know that \( AC = FE \), since \( AC = ryz + rxz = rz(x+y) = FE \). Therefore, \( \triangle CAFE \) is a Saccheri Quadrilateral, and because we are in Euclidean geometry, we can further say that the quadrilateral is a rectangle. We now have that since \( \angle C \) is a right angle, \( \overrightarrow{AC} \perp \overrightarrow{CE} \). Also, since \( \triangle ABCD \) is similar to right triangle 1 in the left figure above, we know that \( \overrightarrow{BC} \perp \overrightarrow{CD} \). Therefore, combining this with Axiom I-1, we know that C, D, and E must be collinear and we can therefore conclude that \( \overrightarrow{FD} \) in triangle 2 and 3 is
indeed the same segment. Note that since C, D, and E are collinear, we can also say that the angle sum at vertex D must equal 180°. Since opposite sides of a rectangle are congruent, we know that in rectangle ACEF, we can equate the opposite side lengths as seen in the figure above and therefore \( xyz = r^2 z + r^2 (x + y) \Rightarrow xyz = r^2 (x + y + z) \). 

Another proof of Lemma 2 is shown in APPENDIX A which involves the use of Heron’s Formula together with Lemma 3 below.

**Lemma 3:** \( K = r(x + y + z) \)

Finally, we examine the area of \( \triangle ABC \) and its relation to the side lengths of the triangle. For this proof, label the angle measures as follows and then rearrange the six triangles from figure 3 to form three rectangles each with height \( r \) (Figure 6 based on Nelsen):

![Figure 6: Equating Areas](image)

It is fairly obvious that the three rectangles are truly rectangles since each one is composed of two of the same triangle flipped over. Therefore, for example, in the first rectangle we know \( a + a + 90 = 180^\circ \Rightarrow a + a = 90^\circ \). Notice that at the two vertices where the angle measure is not given to be 90°, the angle measure is \( a + \alpha \) by the Angle Addition Postulate and therefore each angle in the rectangle does have a measure of 90°.
Adding the areas of the three rectangles together produces \( rx + ry + rz = r(x + y + z) \).

Since these three rectangles are composed of the six triangles that made up \( \Delta ABC \) we can then say that \( K = r(x + y + z) \).

We are now ready to prove Euler’s Triangle Inequality. Starting with the inequality \( abc \geq 8xyz \) we get:

\[
abc \geq 8xyz \\
\Rightarrow 4KR \geq 8xyz \quad \text{Lemma 1} \\
\Rightarrow 4KR \geq 8r^2(x + y + z) \quad \text{Lemma 2} \\
\Rightarrow 4KR \geq 8rK \quad \text{Lemma 3} \\
\Rightarrow R \geq 2r \quad \text{Lemma 2}
\]

Note that \( R = 2r \) if and only if \( x = y = z \) which is true if and only if the triangle is equilateral.

4.2 Feuerbach’s Nine Point Circle

Euler’s Triangle Inequality does not rely on the newer geometric methods being used by the previous mathematicians mentioned, but instead utilizes the postulates and axioms found in the \textit{Elements}. Likewise, another discovery made in 1822 was that of Karl Feuerbach, a German geometer born in 1800. Feuerbach asserted that in any triangle, there are nine points that lie on a unique circle. This circle soon came to be known as the nine point circle. Although it is debated who actually was first in uncovering this remarkable circle, Feuerbach is often credited for the discovery (Smart, 1998). Before anything can be proved concerning the nine point circle, a few definitions and a theorem must be stated.
Definition: An altitude is a perpendicular line from a vertex of a triangle to the line through its opposite side. The point at which an altitude intersects the line through this opposite side is called the foot of the altitude.

Theorem: All three altitudes intersect in a single point called the orthocenter.

Definition: An Euler point is the midpoint of the segment whose endpoints are the orthocenter and a vertex of a triangle. Note that a triangle has three Euler points.

Definition: A medial triangle is a triangle in which the three vertices are midpoints of the sides of a larger triangle. We say that one triangle is the medial triangle of another (the larger one).

Midpoint Connector Theorem: In \( \triangle ABC \), if \( M \) and \( N \) are the midpoints of \( AB \) and \( AC \) respectively, then \( MN \parallel BC \).

Lemma: If \( AB \) is a diameter of a circle and \( m<ACB = 90^\circ \), then \( C \) must lie on that circle.
Proof: If C does not lie on the circle, then C can either lie inside or outside of the circle. Consider when C lies outside of the circle. Construct point D where $\overline{OC}$ intersects the circle. We know that $m \angle ADB = 90^\circ$ by a corollary to the Inscribed Angle Theorem which states that an angle inscribed in a semicircle is a right angle. By the Exterior Angle Inequality and the Angle Addition Postulate, $m \angle ACO + m \angle OCB < m \angle ADO + m \angle ODB = 90^\circ \Rightarrow m \angle ACB < 90^\circ$. The same idea applies to when C lies inside the circle. By the Exterior Angle Inequality and the Angle Addition Postulate, we find that $m \angle ACB > 90^\circ$. Thus, the only way $m \angle ACB = 90^\circ$ when $\overline{AB}$ is the diameter of the circle is for C to lie on the circle.  

Theorem: Given any triangle, all of the following points lie on a common circle; the three feet of the altitudes, the three midpoints of the sides, and the three Euler points. (Isaacs, 2001).
Figure 9: The Nine Points of the Nine Point Circle

In the figure to the left, points D, E, and F are the three feet of the altitudes, points P, Q, and R are the midpoints of each of the three sides of \( \triangle ABC \), points X, Y, and Z are the Euler points, and H is the orthocenter of \( \triangle ABC \). We will prove that each of these nine points lie on the unique circumcircle of the medial \( \triangle APQR \) of \( \triangle ABC \).

**Proof** (based on Isaacs): Construct \( \overline{YQ} \) and a circle such that \( \overline{YQ} \) is a diameter of this circle. To show P is on this circle, we will show that \( \angle YPQ = 90^\circ \). Then by the lemma above, since \( \overline{YQ} \) is a diameter of the circle, P will lie on the circle.

We begin by showing that \( \overline{PQ} \parallel \overline{AB} \). Looking at \( \triangle ABC \), since Q is the midpoint of \( \overline{AC} \) and P is the midpoint of \( \overline{BC} \) we know \( \overline{PQ} \parallel \overline{AB} \). By the same reasoning, we can show that \( \overline{YP} \parallel \overline{CF} \). We apply the M.C.T. this time to \( \triangle BHC \). Since Y is the midpoint of \( \overline{BH} \) and P is the midpoint of \( \overline{BC} \), we have \( \overline{YP} \parallel \overline{CH} \). We want \( \overline{YP} \parallel \overline{CF} \) and since F, H, and C are collinear (they all lie on altitude \( \overline{CF} \)) we obtain this desired result. Finally, we use the fact that \( \overline{AB} \perp \overline{CF} \) to say that \( \overline{YP} \perp \overline{PQ} \) and therefore \( \angle YPQ = 90^\circ \). Hence, by this result we know that P lies on the circle with diameter \( \overline{YQ} \).
We again can use this process to show that R lies on the same circle, this time showing that $\angle QRY = 90^\circ$. To do this, we can show that $\overline{QR} \perp \overline{RY}$, since $\overline{RY} \parallel \overline{AD}$ by applying the M.C.T. to $\triangle ABH$ and $\overline{QR} \parallel \overline{BC}$ by applying the M.C.T. to $\triangle ABC$.

It has been shown that the circle with diameter $\overline{YQ}$ that we began with contains points Q, P, R, and Y. Hence, by definition of circumcircle, this circle is the circumcircle of the medial triangle of $\triangle ABC$.

Next, we construct $\overline{PX}$ as the diameter of a circle. Repeating the same process as above, we can show that R and Q lie on this circle. Therefore, since P, Q, and R all lie on the circle, and the circumcircle of a triangle is unique, we know that this circle is the same as the one above and as a result, $\overline{PX}$ is a diameter of the circumcircle of $\triangle APQR$ and points Y, P, Q, R, and X lie on this circle. Using this argument one more time results in point Z lying on the circumcircle of $\triangle APQR$ where $\overline{RZ}$ is a diameter.

It remains to be shown that points D, E, and F lie on the circumcircle of $\triangle APQR$ as well. To show that point E lies on this circle, we apply our lemma, using the fact that $\angle YEQ = 90^\circ$ by definition of altitude. We repeat this process twice to find that points D and F also lie on the circumcircle. Thus, in $\triangle ABC$, the nine points mentioned above are contained in the circumcircle of the medial triangle of $\triangle ABC$.

**Corollary:** Each of the line segments joining an Euler point to the midpoint of the opposite side (of the original triangle) is a diameter of the nine point circle.
Proof: In the proof above, it was shown that $XP$, $YQ$, and $ZR$ are diameters of the nine point circle.

![Diagram of the Circumcircle of the Medial Triangle of $\triangle ABC$]

Figure 10: The Circumcircle of the Medial Triangle of $\triangle ABC$

Now that we have established that the nine point circle exists, we will concentrate on where the center of the circle lies and what its radius is. The following definition and theorems are needed for later reference:

**Theorem:** All three perpendicular bisectors of the sides of a triangle intersect in a single point called the circumcenter.

**Theorem:** All three medians of a triangle intersect in a single point called the centroid.

**Definition:** The Euler line is the unique line through the circumcenter and the centroid of any non-equilateral triangle. Note that the triangle cannot be equilateral because this would mean that the circumcenter and centroid coincide and therefore no unique line could exist between the two.
Theorem: Assume $\triangle ABC$ is not equilateral and let $G$ be the centroid and $O$ be the circumcenter of the triangle. Let $H$ be the point on the Euler line $GO$ that lies on the opposite side of $G$ from $O$ such that $HG = 2GO$. Then $H$ is the orthocenter of $\triangle ABC$ (Isaacs, 2001) (see APPENDIX B for proof).

We begin looking at $\triangle ABC$ and its medial $\triangle PQR$. Construct medians $\overline{AP}$, $\overline{BQ}$, and $\overline{CR}$. The point of concurrency of the three medians, $G$, is the centroid. Let $T$ be the point of intersection of $\overline{AP}$ and $\overline{BQ}$, $U$ be the intersection of $\overline{BQ}$ and $\overline{RP}$, and $V$ be the intersection of $\overline{CR}$ and $\overline{QP}$.

Figure 11: Parallelogram $AQPR$

We know that $\overline{RP} \parallel \overline{AQ}$ and $\overline{PQ} \parallel \overline{AR}$ by the Midpoint Connector Theorem. Therefore, we have $\triangle AQPR$ is a parallelogram by definition of parallelogram and furthermore know that $AT = TP$ and $RT = TQ$ since the diagonals of a parallelogram bisect each other. Since $T$ is the midpoint of $\overline{RQ}$, then $\overline{PT}$ is a median of $\triangle PQR$. Likewise it can be shown that $\overline{QU}$ and $\overline{RV}$ are also medians of $\triangle PQR$ and hence $G$ is the centroid of $\triangle PQR$ as well.

Now consider the transformation $T$ that is the product of a dilation with scale factor $k = 0.5$ and center $G$ with a rotation of $180^\circ$ about point $G$. By a theorem about the centroid of a triangle (see APPENDIX B), we know that $AG = 2GP$ and since $A$, $T$, and $P$ are collinear, $T(A) = P$. By the same reasoning, $T(B) = Q$ and $T(C) = R$. 

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It can be shown that $T$ carries lines to lines, triangles to triangles, and circles to circles (Kay, 2001). We now look back at the circumcircle of $\triangle ABC$ whose center is $O$ and radius is $K$. Under the transformation $T$, $\triangle ABC$ is carried to $\triangle APQR$ and as a result, the circumcircle of $\triangle ABC$ is carried to the circumcircle of $\triangle APQR$. Remember that the circumcircle of $\triangle APQR$ is the nine point circle. Also, the center of the circumcircle of $\triangle ABC$, $O$, is carried to the center of the nine point circle. Let $N$ be the center of the nine point circle. Then $T(O) = N$ and if the circumcircle of $\triangle ABC$ has radius $K$, then the radius of the nine point circle is $\frac{K}{2}$. We can then say that the diameters of the nine point circle, $\overline{XP}$, $\overline{YQ}$, and $\overline{ZR}$ have length $K$.

Since $\overline{XP}$, $\overline{YQ}$, and $\overline{ZR}$ are all diameters of the nine point circle, it can easily be deduced that $N$ is the midpoint of each of these segments. However, we shall find a way to relate $N$ to the orthocenter, circumcenter, and centroid of $\triangle ABC$.

We know that $O$ and $G$ can either be the same point or distinct. If $O = G$, then $\triangle ABC$ would be equilateral (see APPENDIX C) and then $O = G = H = N$, which tells us where $N$ is located. Therefore, assume $\triangle ABC$ is not equilateral. We then know that the Euler line exists through points $O$ and $G$ in $\triangle ABC$. By definition of $T$, $O$, $G$, and $N$ are collinear with $N-G-O$ and $OG = 2GN \Rightarrow \frac{1}{2}OG = GN$. Also, from the proven theorem above, $H$ lies on the Euler line with $H-G-O$ and $HG = 2GO \Rightarrow \frac{1}{2}HG = GO$. It follows that $H-N-G$ and by definition of betweeness $HG = HN + NG \Rightarrow HN = HG - NG = 2GO - \frac{1}{2}GO = \frac{3}{2}GO$. Also, since $N-G-O$ holds,
\[ NO = NG + GO = \frac{1}{2} GO + GO = \frac{3}{2} GO. \] Hence, \( HN = NO = \frac{3}{2} \) and by definition of midpoint, \( N \) is the midpoint of \( HO \). Thus, we have shown the following:

**Theorem:** The center of the nine point circle of a triangle is the midpoint of the segment connecting the orthocenter and the circumcenter of that triangle.

\[ \blacksquare \]

**Figure 12:** The Position of \( N, H, O, \) and \( G \) on the Euler Line

4.3 Morley's Theorem

Moving away from circles and looking solely at triangles, we now turn to a discovery of the mathematician Frank Morley in 1899. Morley was born in 1860 in Woodbridge, England. After suffering from poor health in England, he moved to the United States and became a professor of mathematics. Although he is not credited with multiple mathematical theorems in geometry, he did make the following important discovery.
Morley's Theorem: The adjacent trisectors of the angles of a triangle are concurrent by pairs at the vertices of an equilateral triangle (Smart, 1998). Thus in the following figure, if the angles at vertices A, B, and C are trisected, then ΔXYZ is equilateral.

![Figure 13: Equilateral ΔXYZ](image)

In order to prove Morley's Theorem, the following lemma is needed:

**Lemma:** Let $\overline{AM}$ be the bisector of $\angle BAC$ where M lies on $\overline{BC}$. Then the incenter of ΔABC is the unique point L on $\overline{AM}$ such that $m\angle BLC = 90 + \frac{1}{2} m\angle BAC$.

![Figure 14: A Property of the Incenter](image)

**Proof:** We will first assume that L is the incenter and show that $m\angle BLC = 90 + \frac{1}{2} m\angle BAC$. Then we will show that only when L is the incenter is this equation true.
Part 1 (Proof based on Isaacs): Let $L$ be the incenter of $\Delta ABC$. Then, by the definition of incenter, $L$ lies on the angle bisectors of $\Delta ABC$. Looking at $\Delta BLC$, we see that:

$$m_{BLC} = 180 - m_{LBC} - m_{LCB}$$

$$= 180 - \frac{1}{2}m_{ABC} - \frac{1}{2}m_{ACB}$$

$$= 180 - \frac{1}{2}(m_{ABC} + m_{ACB})$$

$$= 180 - \frac{1}{2}(180 - m_{BAC})$$

$$= 90 + \frac{1}{2}m_{BAC}$$

Part 2: Let $L$ be the incenter of $\Delta ABC$. Then $m_{BLC} = 90 + m_{BAC}$. Construct $N$ such that $A-N-L$ holds. By the Exterior Angle Inequality and the Angle Addition Postulate, we know $m_{BLN} + m_{LNC} < m_{BLM} + m_{LMC} = 90 + \frac{1}{2}m_{BAC}$

$$\Rightarrow m_{BNC} < 90 + \frac{1}{2}m_{BAC}.$$ Now consider when $N$ is constructed on $\overline{AM}$ such that $L-N-M$ holds. Using the same reasoning as above, we know that

$$m_{BNM} + m_{MNC} < m_{BLM} + m_{MLC} = 90 + \frac{1}{2}m_{BAC}$$

$$\Rightarrow m_{BNC} < 90 + \frac{1}{2}m_{BAC}.$$ Therefore, the incenter $L$ is the only point on $\overline{AM}$ such that $m_{BLC} = 90 + \frac{1}{2}m_{BAC}$. $\blacksquare$

Proof of Morley's Theorem (based on Isaacs): Let $\Delta UVW$ be given and $\Delta XYZ$ be equilateral. We will construct a larger $\Delta ABC$ that is similar to $\Delta UVW$ such that its
adjacent angle trisectors meet at points X, Y, and Z and therefore form a Morley configuration as stated in the theorem. Hence Morley's theorem will be satisfied since corresponding angles are congruent in similar triangles.

Let \( \alpha = \frac{1}{3} m \angle U \Rightarrow 3 \alpha = m \angle U \), \( \beta = \frac{1}{3} m \angle B \Rightarrow 3 \beta = m \angle V \), and \( \gamma = \frac{1}{3} m \angle W \)

\( \Rightarrow 3 \gamma = m \angle W \). Then:

\[
\alpha + \beta + \gamma = \frac{1}{3} m \angle U + \frac{1}{3} m \angle V + \frac{1}{3} m \angle W
\]

\[
= \frac{1}{3} (m \angle U + m \angle V + m \angle W)
\]

\[
= \frac{1}{3} (180)
\]

\[
= 60
\]

Construct point P on the opposite side of \( \overline{ZY} \) as X such that

\( m \angle PZY = m \angle PYZ = \beta + \gamma \). Construct point Q on the opposite side of \( \overline{ZX} \) as Y such that \( m \angle QXZ = m \angle QZX = \alpha + \gamma \). Construct point R on the opposite side of \( \overline{XY} \) as Z such that \( m \angle RXY = m \angle RYX = \alpha + \beta \).
Figure 15: Constructing ΔABC

Extend \( OZ \) in the direction of \( Z \) and \( RY \) in the direction of \( Y \). Then \( OZ \) and \( RY \) will meet at some point \( A \). To see why this is true, consider \( \theta = m\angle QZY + m\angle RYZ \).

Observe that \( OZ \) and \( RY \) will meet on the same side of \( ZY \) as \( P \) only if \( \theta > 180^\circ \). If \( \theta = 180^\circ \), then by Property C (Kay, 2001) \( OZ \parallel RY \) and hence the segments will never meet. If \( \theta < 180^\circ \), then \( OZ \) and \( RY \) will meet, but on the opposite side of \( ZY \) as \( P \).

We know \( m\angle QZY = m\angle QZX + m\angle XZY \) by the Angle Addition Postulate and likewise \( m\angle RYZ = m\angle RYX + m\angle XYZ \). Therefore:

\[
\theta = m\angle QZY + m\angle RYZ
\]

\[
= (m\angle QZX + m\angle XZY) + (m\angle RYX + m\angle XYZ)
\]

\[
= (\alpha + \gamma + 60) + (\alpha + \beta + 60)
\]

\[
= (\alpha + \beta + \gamma) + \alpha + 120
\]

\[
= 60 + \alpha + 120
\]

\[
= 180 + \alpha
\]
Since $\theta = 180 + \alpha$ and $\alpha > 0$ because $\alpha$ is an angle measure, we know $\theta > 180$ and therefore $\overline{QZ}$ meets $\overline{RY}$ at point $A$ on the same side of $\overline{ZY}$ as $P$.

Now consider $\triangle AZY$. We know:

$$m\angle ZAY = 180 - m\angle AZY - m\angle AYZ$$

$$= 180 - (180 - m\angle QZY) - (180 - m\angle RYZ)$$

$$= m\angle QZY + m\angle RYZ - 180$$

$$= 180 + \alpha - 180$$

$$= \alpha$$

Repeating this process, we find that if we extend $\overline{PZ}$ and $\overline{RX}$ they will meet at $B$ where $m\angle XBZ = \beta$ and if we extend $\overline{QX}$ and $\overline{PY}$ they will meet at $C$ and the $m\angle YCX = \gamma$.

Figure 16: Finding $\alpha$, $\beta$, and $\gamma$
We have $\overline{ZX} \equiv \overline{ZY}$, $\overline{YR} \equiv \overline{XR}$, and $\overline{RZ} \equiv \overline{RZ}$ so by SSS, $\Delta ZYR \equiv \Delta ZXR$. By CPCF, $\angle ZRX \equiv \angle ZRY$. By definition of angle bisector, $\overline{ZR}$ bisects $\angle ARB$ which implies $Z$ lies on the angle bisector of $\angle ARB$. We now want to show that 

$m \angle AZB = 90 + \frac{1}{2} m \angle ARB$ which implies that $Z$ is the incenter of $\Delta ARB$ by the lemma above.

$m \angle AZB = m \angle PZQ$ \hspace{1cm} \text{Vertical Pair Theorem}

\begin{align*}
= & \ m \angle PZY + m \angle YZX + m \angle XZQ \\
= & \ (\beta + \gamma) + 60 + (\alpha + \gamma) \\
= & \ 60 + 60 + \gamma \\
= & \ 120 + \gamma
\end{align*}

Now looking at $\Delta YRX$, we know that:

$m \angle YRX = 180 - m \angle RYX - M \angle RXY$

\begin{align*}
= & \ 180 - 2(\alpha + \beta) \\
= & \ 180 - 2(60 - \gamma) \\
= & \ 60 + 2\gamma
\end{align*}

Note that $\angle ARB = \angle YRX$ since $A, Y,$ and $R$ are collinear and $B, X,$ and $R$ are collinear. Then:

\begin{align*}
90 + \frac{1}{2} m \angle ARB & = 90 + \frac{1}{2} (60 + 2 \gamma) \\
& = 120 + \gamma \\
& = m \angle AZB
\end{align*}
Since \( m\angle AZB = 90 + \frac{1}{2} m\angle ARB \), we have shown that \( Z \) is the incenter of \( \triangle ARB \) which implies that \( \overline{AZ} \) bisects \( \angle BAR \) and hence \( m\angle BAZ = m\angle ZAY = \alpha \). By the same reasoning we can find that \( Y \) is the incenter of \( \triangle AQC \) which implies \( m\angle QAY = m\angle YAC = \alpha \). Therefore, \( m\angle BAC = 3\alpha \) by the Angle Addition Postulate and \( \overline{AZ} \) and \( \overline{AY} \) are the angle trisectors of \( \angle BAC \). Similarly, we can find that \( m\angle ACB = 3\gamma \) where \( \overline{CY} \) and \( \overline{CX} \) are the angle trisectors of \( \angle ACB \) and \( m\angle CBA = 3\beta \) where \( \overline{BX} \) and \( \overline{BZ} \) are the angle trisectors of \( \angle ABC \). Thus we have \( \triangle ABC \sim \triangle UVW \) by the AA Similarity Criterion and we have shown that given any triangle, the adjacent angle trisectors meet at three points which form an equilateral triangle.
5. THE GEOMETRY OF TODAY

Not only has geometry evolved and been studied over thousands of years, but it also is the basis for several other fields that have rapidly developed over the last one hundred years. One such field is topology which became a subject of great interest in the 20th century. The word topology is derived from the Greek words "topos" and "logos" which mean "place" and "study" respectively. Topology was founded by Henri Poincaré, a French mathematician who is considered to be the father of topology, and who formulated the Poincaré conjecture which was not solved until 2002. The actual word "topology" was first used in 1847 by Joseph Listing, a pupil of Carl Gauss, in his textbook *Verstudeien zur Topologie* (Burton, 1997, p. 642). The field builds on set theory and involves "the mathematical study of the properties that are presevered through deformations, twistings, and stretchings of objects" (Weisstein). One idea that is studied in topology is the concept of objects being topologically equivalent to one another. Two objects are topologically equivalent if one can be made to look like the other without tearing or breaking the object. For example, a circle is topologically equivalent to an ellipse since a circle can be stretched to form an ellipse.

Another important branch of geometry is algebraic geometry which was also heavily developed in the 20th century. As its name implies, algebraic geometry "is the study of geometries that come from algebra," where "the algebra is the ring of polynomials and the geometry is the set of zeros of the polynomials called an algebraic variety" (Rowland). This field has been extremely important to finding solutions to
problems in which no solutions could be found for centuries. One such problem is
Fermat's Last Theorem which states that one “can never find whole numbers \( a, b, \) and \( c, \)
and an exponent \( n \geq 3 \) for which \( a^n + b^n = c^n \) (Dunham, 1997, p. 159). In 1995, Andrew
Wiles proved this conjecture using tools from algebraic geometry and thus ended the
question of whether or not Fermat's conjecture could be proven.

There are many other branches of geometry that mathematicians continue to study
today such as differential and projective geometry. The list does not stop here, however,
for there are multiple applications and places where one can find some aspect of the
subject in whatever they may be studying. The importance of geometry to everyone is
shown by its presence in classrooms around the world and the topics taught in this area
are those which will continue to be instilled in some facet of every person's life for
centuries to come.
REFERENCES


APPENDIX A

Alternate Proof of Lemma 2: $xyz = r^2 (x + y + z)$

Another proof of Lemma 2 can be obtained by finding the area of $\Delta ABC$ using Heron’s Formula and then equating this result with the area of $\Delta ABC$ found in Lemma 3.

Heron’s Formula: Given $\Delta ABC$ and the lengths of its sides $a, b, c$, the area of the triangle is equal to $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2} (a+b+c)$.

We first find $s$ using the fact that since $a = y + z, b = z + x, and c = x + y$ we have $s = \frac{1}{2} (a+b+c) = \frac{1}{2} [(y+z) + (z+x) + (x+y)] = (x+y+z)$. Now letting $K$ denote the area of $\Delta ABC$, we get:

\[ K = \sqrt{s(s-a)(s-b)(s-c)} \]
\[ = \sqrt{(x+y+z)(x+y+z-(y+z))(x+y+z-(z+x))(x+y+z-(x+y))} \]
\[ = \sqrt{(x+y+z)(x)(y)(z)} \]
\[ = \sqrt{(x+y+z)(xyz)} \]

From Lemma 3, we get that $K = r(x+y+z)$ so equating these two areas of $\Delta ABC$ yields:

\[ \sqrt{(x+y+z)(xyz)} = r(x+y+z) \]
\[ \Rightarrow \left( \sqrt{(x+y+z)(xyz)} \right)^2 = (r(x+y+z))^2 \]
\[ \Rightarrow (x+y+z)(xyz) = r^2 (x+y+z)^2 \]
\[ \Rightarrow (xyz) = r^2 (x+y+z) \quad \blacksquare \]
APPENDIX B

Theorem: Assume \( \triangle ABC \) is not equilateral and let \( G \) be the centroid and \( O \) be the circumcenter of the triangle. Let \( H \) be the point on the Euler line \( GO \) that lies on the opposite side of \( G \) from \( O \) such that \( HG = 2GO \). Then \( H \) is the orthocenter of \( \triangle ABC \) (Isaacs, 2001).

\[
\begin{align*}
\text{Figure 17: The Orthocenter and the Euler Line} \\
\text{Proof (based on Isaacs): Let } \triangle ABC \text{ be a non-equilateral triangle. We will prove that the altitude from } A \text{ contains point } H \text{ and by similar reasoning it can be shown that the altitudes from } B \text{ and } C \text{ also contain } H. \text{ We know it is possible that } H \text{ can coincide with one of the vertices of } \triangle ABC. \text{ However, } H \text{ can only coincide with one vertex. Without loss of generality, assume that } A \text{ and } H \text{ are distinct. It suffices to show that } \overrightarrow{AH} \perp \overrightarrow{BC}. \\
\text{Construct } M \text{ such that } BM = MC. \text{ Using a theorem about the centroid (Isaacs, 2001), we know that the centroid of a triangle lies on a median } \frac{2}{3} \text{ of the way from a}
\end{align*}
\]
vertex of the triangle toward the midpoint of the opposite side. Therefore, the points O
and M must be distinct. If they were not, median \( \overline{AM} \) would lie on the Euler line and by
the above theorem \( AG = 2GM \implies AG = 2GO \). However, we are given that H is a point
on the Euler line that lies on the opposite side of G from O such that \( HG = 2GO \) and
hence, A and H would be the same point, but we have assumed they were distinct.

We know M lies on the perpendicular bisector of \( BC \) since M is the
midpoint of \( BC \), and by definition of circumcenter and the perpendicular bisector
theorem, O lies on the perpendicular bisector of \( BC \) as well. Hence, \( \overline{OM} \perp \overline{BC} \). Note
that \( AG = 2GM \implies \frac{AG}{GM} = 2 \) and \( HG = 2GO \implies \frac{HG}{GO} = 2 \). Also, since vertical angles are
congruent by the vertical pair theorem, \( \angle AGH \equiv \angle MGO \). Therefore, by the SAS
similarity criterion \( \triangle AGH \sim \triangle MGO \) and by definition of similar triangles,
\( \angle AHG \equiv \angle MOG \). Since alternate interior angles, \( \angle AHG \) and \( \angle MOG \), are congruent, it
follows that \( \overline{AH} \parallel \overline{OM} \). Also, we know that since \( \overline{AH} \parallel \overline{OM} \) and \( \overline{OM} \perp \overline{BC} \),
\( \overline{AH} \perp \overline{BC} \).

It is important to note that if H does coincide with one of the vertices of the
triangle, then by the same reasoning as above, it can be shown that H lies on a second
altitude of the triangle (the second altitude that does not contain the vertex that coincides
with H). However, by a previously mentioned theorem, since all three altitudes are
concurrent, we know that if H is the point of intersection of two altitudes then it must be
on the third altitude as well. Thus, H lies on the three altitudes of \( \triangle ABC \) and is the
orthocenter. \( \square \)
APPENDIX C

Theorem: If $O = G$, then $\triangle ABC$ must be equilateral.

![Diagram showing the circumcenter and centroid coincide]

**Figure 18: The Circumcenter and the Centroid Coincide**

**Proof:** Let the circumcenter, $O$, coincide with the centroid, $G$. First, we must show that each median in the triangle lies on the same line as each corresponding perpendicular bisector. Let $I$ be the midpoint of $BC$, $J$ be the midpoint of $AC$, and $L$ be the midpoint of $AB$. We know the perpendicular bisector of $BC$ goes through points $O$ (which equals $G$) and $I$. Also, the median that connects vertex $A$ to the midpoint of $BC$ goes through points $G$ (which equals $O$) and $I$. By one of the incidence axioms we know that two points determine a unique line and therefore, since the perpendicular bisector of $BC$ and the median through $BC$ both contain points $O = G$ and $I$, then the two lines coincide. The same is true of the other two perpendicular bisectors and medians.
Since I is the midpoint of $BC$ we know that $BI = IC$ by definition of midpoint.

Also since $AI$ is the perpendicular bisector of $BC$ we know that $m\angle AIB = m\angle AIC = 90^\circ$.

By the SAS axiom, $\triangle AIB \cong \triangle AIC$. By CPCF, $AB \cong AC$. Using the same reasoning, we can show that $\triangle BJC \cong \triangle BJA$ and therefore by CPCF we know that $BC \cong BA$. Thus, by the transitive property, $BC \cong AC$ and $\triangle ABC$ is equilateral by definition of equilateral triangle.